Multiprime Strategies for Serial Evaluations of eSIDH-Like Isogenies

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Diffie-Hellman Key Establishment

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$$\begin{array}{ccc}
x \xleftarrow{\$} \{0, 1, \ldots, p-1\} & & y \xleftarrow{\$} \{0, 1, \ldots, p-1\} \\
X \leftarrow g^x & & Y \leftarrow g^y \\
& \xrightarrow{\quad X \quad} & \\
& \xleftarrow[\quad Y \quad]{} & \\
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Security: In the quantum setting, **completely broken** by Shor's algorithm. We need new primitives.

Elliptic Curves

For my purposes, an **elliptic curve** is a set of the form

$$E_C/k = \{(x, y) \in \overline{k}^2 : y^2 = x^3 + Cx^2 + x\} \sqcup \{\infty\}$$

for some $C \in k \setminus \{2, -2\}$. This is **Montgomery form**, and C is the **Montgomery coefficient.**

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I will also care about the k-rational points of a curve:

$$E_C(k) = \{(x, y) \in k^2 \,:\, y^2 = x^3 + Cx^2 + x\} \sqcup \{\infty\}.$$

Elliptic Curves and the Group Law

Here's (the real points of) an elliptic curve:

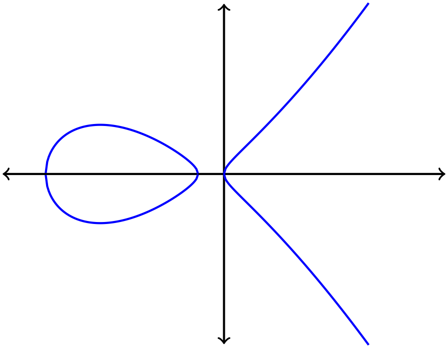
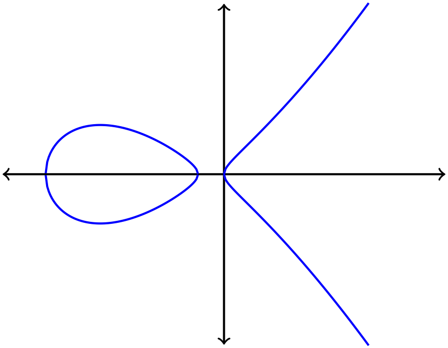


Figure: The curve $E_3/\mathbb{R}\colon y^2 = x^3 + 3x^2 + x$.

Elliptic Curves and the Group Law

Every elliptic curve is also a group, using the chord and tangent law.



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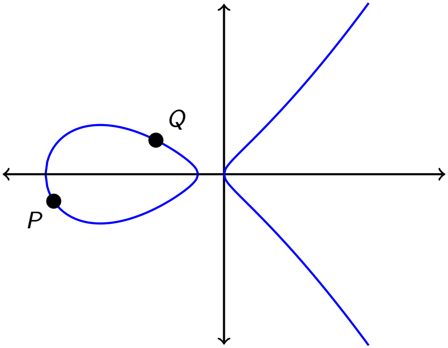

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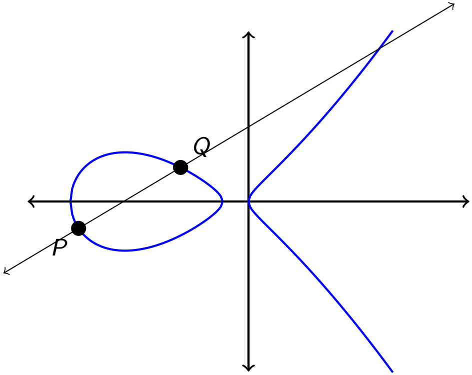

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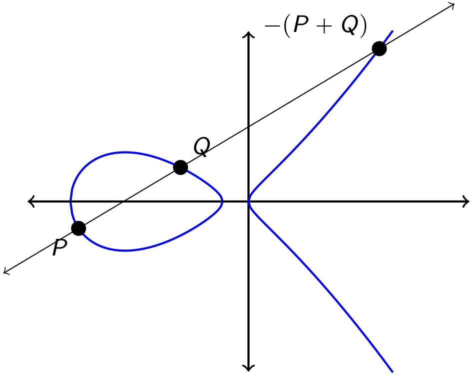

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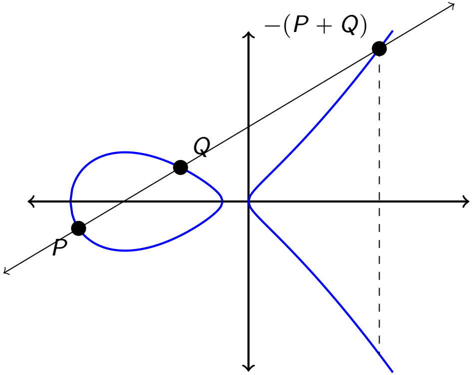

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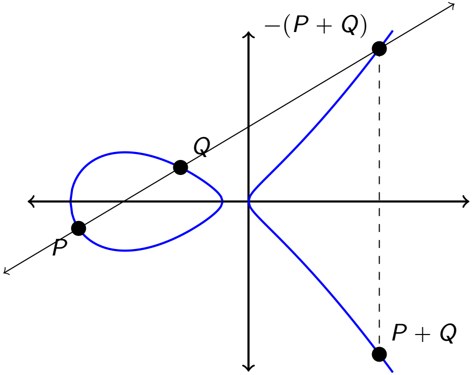

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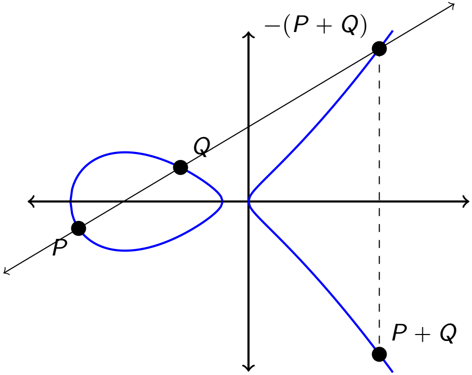
Elliptic Curves and the Group Law

With this group operation in mind, I define the m-**torsion subgroup** of an elliptic curve as

$$E[m] = \{P \in E : [m]P = \infty\}.$$

(here $[m]$ is the multiplication-by-m map).



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In classical cryptography, supersingular elliptic curves are not used as platform groups for Diffie-Hellman, since the discrete logarithm problem can be solved on such curves in subexponential time.

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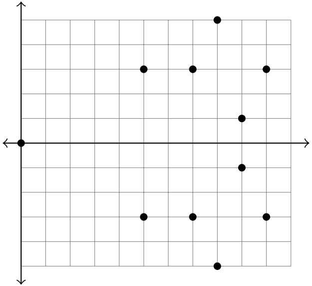
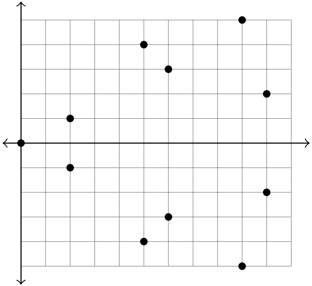
Example: An Isogeny

These curves are related by the isogeny

$$\psi(x,y) = \left(\frac{3x^3+x^2+x}{x^2+x+3},\; y\frac{4x^3-5x^2-3}{x^3-4x^2-2x-4}\right)$$



$E_0/\mathbb{F}_{11} : y^2 = x^3 + x$



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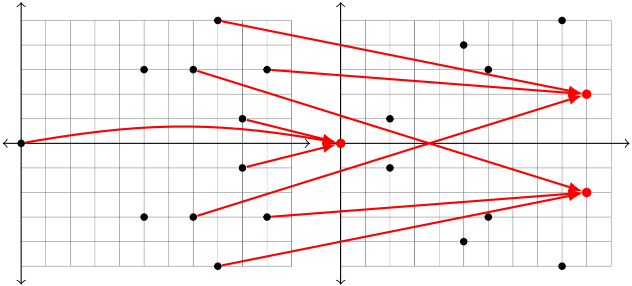

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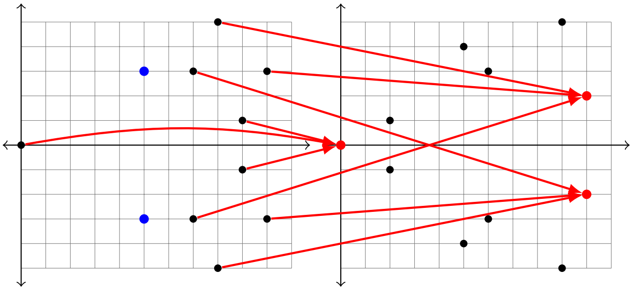
We have $\deg\psi = 3$, so as you expect it is a 3-to-1 map. Its kernel is

$$\ker\psi = \{\infty, (5,3), (5,8)\} = \langle(5,3)\rangle\,.$$



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SIDH: Protocol Description

Global parameters:

- A prime $p = \ell_A^{e_A} \ell_B^{e_B} - 1$ for small primes $\ell_A \neq \ell_B$;
- A supersingular elliptic curve $E/\mathbb{F}_{p^2}$ with $|E(\mathbb{F}_{p^2})| = (p+1)^2$; and,
- Torsion bases $\{P_A, Q_A\} \subseteq E(\mathbb{F}_{p^2})$ for $E[\ell_A^{e_A}]$, $\{P_B, Q_B\} \subseteq E(\mathbb{F}_{p^2})$ for $E[\ell_B^{e_B}]$.

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This only works because E is supersingular and because of the special form of p.

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whose kernels are given by

$$\ker \psi_{A,i} = \langle \overbrace{[\ell_A^{e_A - i}] \cdot \psi_{A,i-1} \circ \cdots \circ \psi_{A,1}(P_A + xQ_A)}^{Q_{A,i}} \rangle$$

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Running SIDH quickly $\iff$ finding the $Q_{A,i}$ quickly.

A Common Thread: Strategies

- There is an obvious way for Alice to compute her $Q_{A,i}$:
  - Set $R_A = P_A + xQ_A$
  - In the i^{th} round:
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- We can represent this pictorially:
  - Vertices $\iff$ Points on curves
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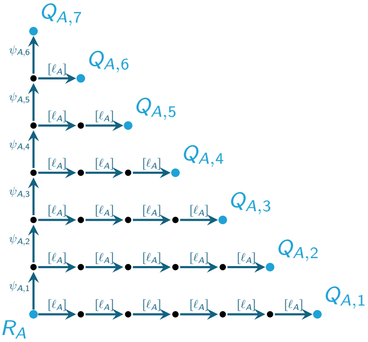

Figure: Some illustrations with $e_A = 7$.

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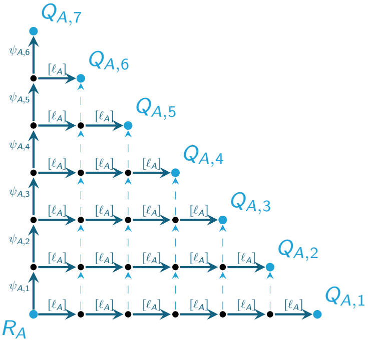
- There are some "unused edges":



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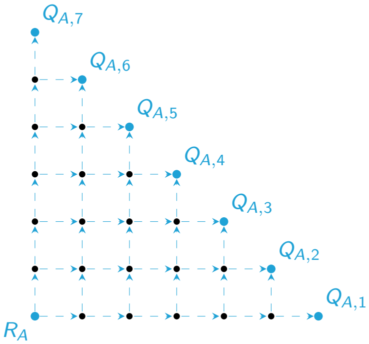

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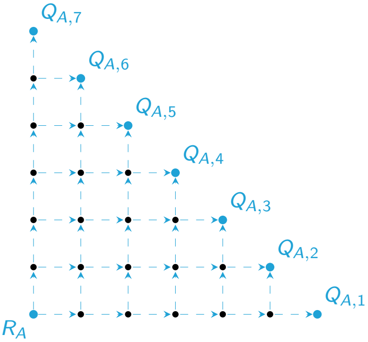
- Any **Steiner arborescence** with root R_A and terminals $\{Q_{A,1}, \ldots, Q_{A,e_A}\}$ gives us an algorithm by following its edges from bottom to top, left to right.
  (A Steiner arborescence in G with root r and terminals T is a connected subgraph of G that contains a path from r to each element of T, and has no undirected cycles).



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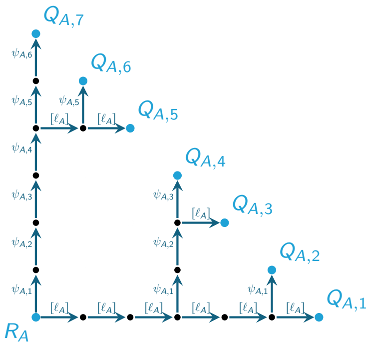
- We call such an arborescence a **strategy**. Here's one:



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+ Corresponds "directly" to running time
- Algorithm must be implemented first!
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For `SIKEp434` (*i.e.*, for the prime $p = 2^{216}3^{137} - 1$):

- Multiplication by 3 takes 2965 cycles/11 field multiplications
- 3-isogeny evaluation takes 1478 cycles/5.6 field multiplications

Strategies in Weighted Graphs for SIDH

We can redraw the graph so that it better depicts the algorithm's cost; in this drawing, the total length of solid edges is (a proxy for) running time:

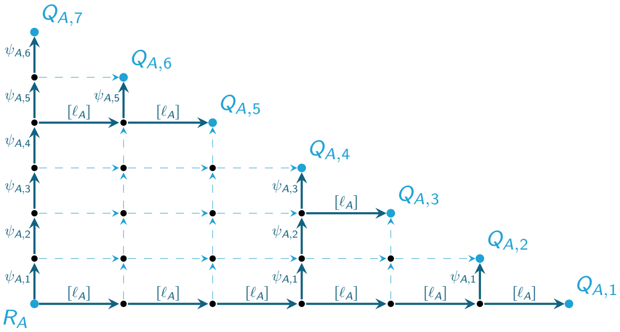

Figure: A weighted strategy for $\ell_A = 3$ and $e_A = 7$.

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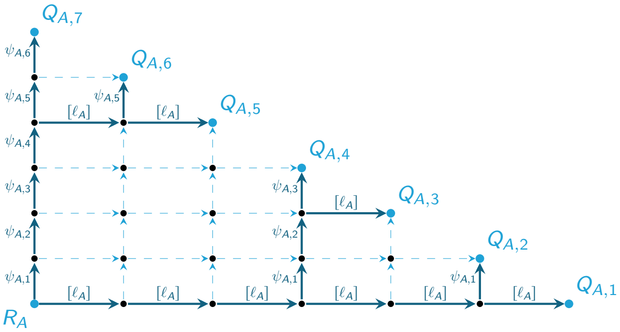

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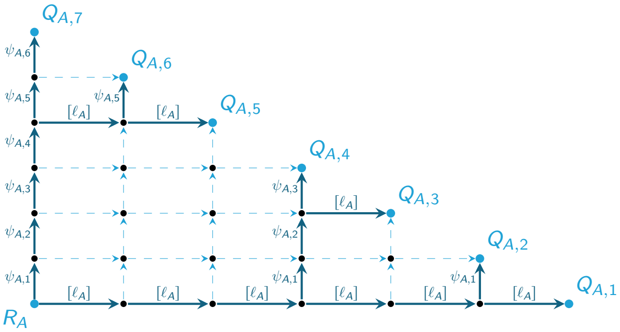
De Feo–Jao–Plût (2011) construct optimal strategies for SIDH using a recursive decomposition/dynamic programming technique.



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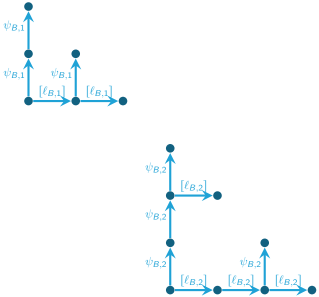

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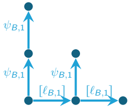
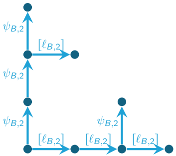


But each "small" strategy requires a torsion basis to construct the root... So in the *second* round, these must be constructed using costly scalar multiplications.



Multiprime Strategies in Serial eSIDH

Note that if $E[\ell_{B,1}^{e_{B,1}}] = \langle P_1, Q_1 \rangle$ and $E[\ell_{B,2}^{e_{B,2}}] = \langle P_2, Q_2 \rangle$ then for any β_1, β_2 there exists an integer β^* such that

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I don't just need to optimize the strategy, but the **permutation** too.

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The reality: for fixed $\vec{\ell}$,

- Given Σ, we can find the optimal S (dynamic programming *à la* De Feo-Jao-Plût);
- Given S, we can find the optimal Σ (linear programming);
- Stochastic search yields (Σ, S) which improves upon the state-of-the-art.

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We can drop $\Sigma \in \mathbb{Z}^{n \times n}$ because the feasible polytope is integral. Then solve an LP for Σ.

eSIDH Timing Estimates

Scheme	Operation	Timings (Mcycles)		Improvement
		Split Prime	Multiprime	
$p_{443} = 2^{222}3^{73}5^{45} - 1$				
eSIDH	Bob R1	7.44	7.75	-4.03%
	Bob R2	**7.00**	**6.47**	**8.24%**
eSIKE	Keygen	7.43	7.74	-4.01%
	Decap	**13.71**	**13.17**	**4.10%**
$p_{765} = 2^{391}3^{119}5^{81} - 1$				
eSIDH	Bob R1	27.14	28.37	-4.34%
	Bob R2	**25.56**	**23.90**	**6.96%**
eSIKE	Keygen	27.14	28.39	-4.42%
	Decap	**50.47**	**48.83**	**3.36%**

Table: eSIDH/eSIKE timing results on Intel i7-8650u processor.