BME 5742 Bio-Systems Modeling and Control

Lecture 12
Linearization Examples 1
Linearization of the Predator-Prey Model

Example 1: Linearization of the basic Predator-Prey Model

$$\frac{dx}{dt} = rx - axy$$

$$\frac{dy}{dt} = -my + bxy$$

- Prey population = x
- Predator population = y
- Positive constants r,m,a,b.

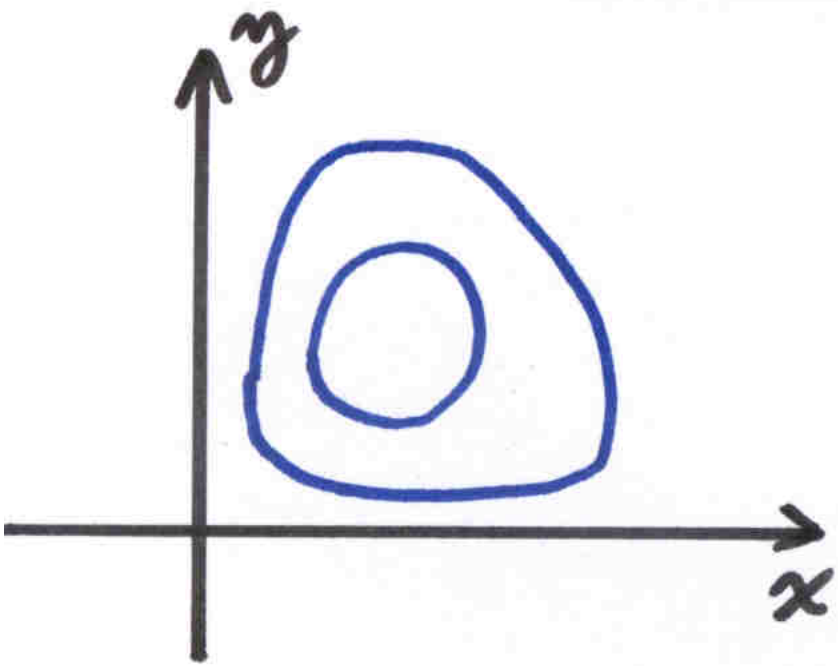
Example 1: Predator-Prey Model Equilibrium States

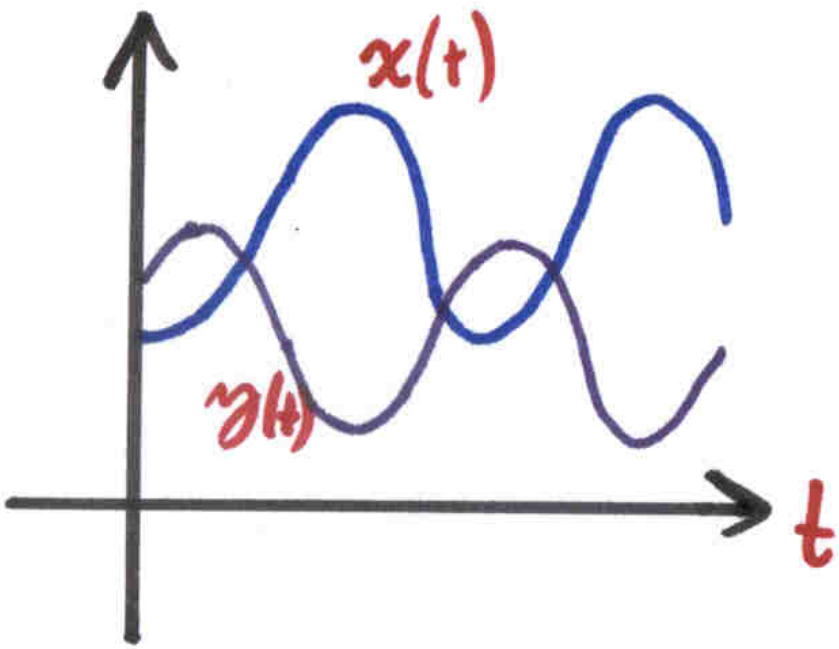
$$\frac{dx}{dt} = 0 = rx - axy$$

$$\frac{dy}{dt} = 0 = -my + bxy$$

There are two solutions x=y=0 and (x=m/b and y = r/a).

Example 1: Recall Predator-Prey State-Plane and Time Solutions





Example 1: Recall the Normalization of the Predator-Prey Model

$$u = \frac{b}{m} x \qquad v = \frac{a}{r} y \qquad \alpha = \frac{m}{r} \qquad \tau = r \cdot t$$

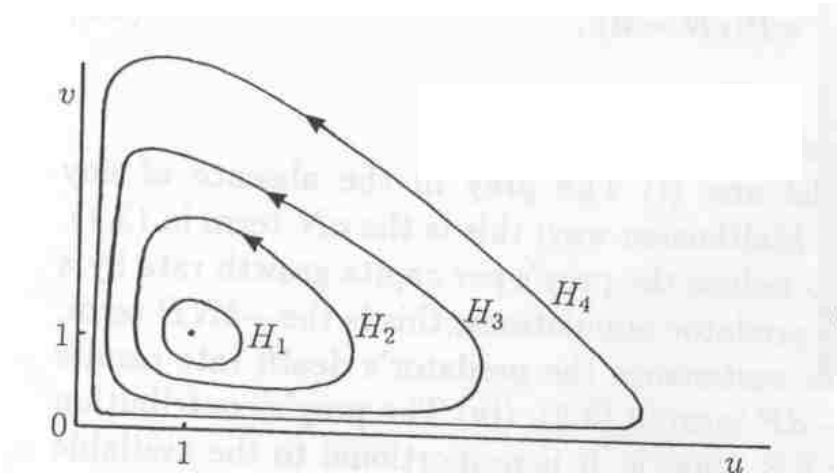
Example 1: Recall the normalized Predator-Prey Model:

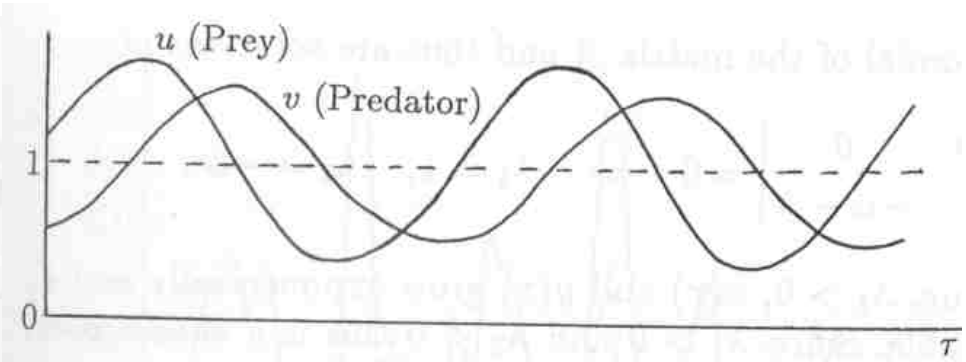
$$\frac{du}{d\tau} = u(1-v) \qquad \frac{dv}{d\tau} = \alpha v(u-1)$$

- Equilibrium states are u=v=0 and u=v=1

Example 1: Recall the resulting normalized Predator-Prey Model and solution:

$$\frac{du}{d\tau} = u(1-v) \qquad \frac{dv}{d\tau} = \alpha v(u-1)$$





Example 1: Finding the A matrix

$$\frac{du}{d\tau} = u(1-v) = f_1(u,v) \qquad \frac{dv}{d\tau} = \alpha v(u-1) = f_2(u,v)$$

$$x = (u,v)^T, A = \frac{\partial f}{\partial x}\Big|_{x_e} = \begin{pmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \end{pmatrix}\Bigg|_{x_e} = \begin{pmatrix} 1-v & -u \\ \alpha v & \alpha(u-1) \end{pmatrix}\Bigg|_{x_e}$$

Example 1: Stability of $(0,0)^T$

$$\frac{du}{d\tau} = u(1-v) = f_1(u,v) \qquad \frac{dv}{d\tau} = \alpha v(u-1) = f_2(u,v)$$

$$x = (u,v)^T, A = \frac{\partial f}{\partial x}\Big|_{x_e} = \begin{pmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \end{pmatrix}\Bigg|_{x_e} = \begin{pmatrix} 1-v & -u \\ \alpha v & \alpha(u-1) \end{pmatrix}\Bigg|_{x_e}$$

$$x_{e1} = (0,0)^T, A = \frac{\partial f}{\partial x}\Big|_{x_{e1}} = \begin{pmatrix} 1-v & -u \\ \alpha v & \alpha(u-1) \end{pmatrix}\Bigg|_{x_{e1}} = \begin{pmatrix} 1 & 0 \\ 0 & -\alpha \end{pmatrix}$$

$$\det\begin{pmatrix} \lambda-1 & 0 \\ 0 & \lambda+\alpha \end{pmatrix} = 0 = (\lambda-1)(\lambda+\alpha)$$

$(0,0)^T$ is unstable

Example 1: Stability of $(1,1)^T$

$$\frac{du}{d\tau} = u(1-v) = f_1(u,v) \qquad \frac{dv}{d\tau} = \alpha v(u-1) = f_2(u,v)$$

$$x = (u,v)^T, A = \frac{\partial f}{\partial x}\Big|_{x_e} = \begin{pmatrix} \frac{\partial f_1}{\partial u} & \frac{\partial f_1}{\partial v} \\ \frac{\partial f_2}{\partial u} & \frac{\partial f_2}{\partial v} \end{pmatrix}\Bigg|_{x_e} = \begin{pmatrix} 1-v & -u \\ \alpha v & \alpha(u-1) \end{pmatrix}\Bigg|_{x_e}$$

$$x_{e2} = (1,1)^T, A = \frac{\partial f}{\partial x}\Big|_{x_{e2}} = \begin{pmatrix} 1-v & -u \\ \alpha v & \alpha(u-1) \end{pmatrix}\Bigg|_{x_{e2}} = \begin{pmatrix} 0 & -1 \\ \alpha & 0 \end{pmatrix}$$

$$\det\begin{pmatrix} \lambda & 1 \\ \alpha & \lambda \end{pmatrix} = 0 = \lambda^2 + \alpha \rightarrow \lambda_{1,2} = \pm j\sqrt{\alpha}$$

Stability of $(1,1)^T$ cannot be determined. It may be a center