BME 5742 Bio-Systems Modeling and Control

Lecture 6

Logistic Model with Time Delay

Deficiency of all models shown thus far:

- Birth rate was considered to act instantly, based on the present population size.
- For many species there is a considerable time delay (pregnancy, juveniles reaching maturity etc.)
- A more reasonable model should assume that present birth rate (at time t) actually depends on $y(t-T_d)$ and not on $y(t)$, where T_d is a known pure time delay.

Delay may cause an oscillatory behavior

Example: (simple linear equation)

$$\frac{dy(t)}{dt} = -\frac{\pi}{2T_d} y(t - T_d) \Rightarrow$$

$$\Rightarrow y(t) = A\cos(\frac{\pi \cdot t}{2T_d})$$

Above can be verified by direct substitution

Logistic Delay Model

$$\frac{dy(t)}{dt} = ry(t)[1 - \frac{y(t - T_d)}{K}]$$

In this model the regulatory effect depends on the population at an earlier time

Birth effect (here) is not delayed.

Logistic Delay Model Normalized

$$\frac{dy(t)}{dt} = ry(t)[1 - \frac{y(t - T_d)}{K}]$$

$$x = \frac{y}{K} \qquad \tau = rt \qquad D = rT_d$$

$$\frac{dx(\tau)}{d\tau} = x(\tau)[1 - x(\tau - D)]$$

For the time being we are not discussing the mathematical details of the scaling process

Logistic Delay Model : Known Fact I

$$\frac{dy(t)}{dt} = ry(t)[1 - \frac{y(t-T_d)}{K}] \quad x = \frac{y}{K} \qquad \tau = rt \qquad D = rT_d$$

$$\frac{dx(\tau)}{d\tau} = x(\tau)[1 - x(\tau - D)]$$

If $D=rT_d < \pi/2$ there is no limit cycle (i.e. a steady oscillation). Population y goes to K (or equivalently x goes to 1).

Logistic Delay Model : Known Fact II

$$\frac{dy(t)}{dt} = ry(t)[1 - \frac{y(t-T_d)}{K}] \quad x = \frac{y}{K} \qquad \tau = rt \qquad D = rT_d$$

$$\frac{dx(\tau)}{d\tau} = x(\tau)[1 - x(\tau - D)]$$

If $D=rT_d > \pi/2$ there is a limit cycle (i.e. sustained oscillation) about y=K with a period T which is in the order of magnitude of $T \approx 4T_d$

Logistic Delay Model : Known Fact III

$$\frac{dy(t)}{dt} = ry(t)[1 - \frac{y(t-T_d)}{K}] \quad x = \frac{y}{K} \quad \tau = rt \quad D = rT_d$$

$$\frac{dx(\tau)}{d\tau} = x(\tau)[1 - x(\tau - D)]$$

Example 1: $D=rT_d = 1.6$ then: Oscillation period $T \approx 4.03T_d$ and $y_{max}/y_{min} \approx 2.56$

Logistic Delay Model : Known Fact IV

$$\frac{dy(t)}{dt} = ry(t)[1 - \frac{y(t-T_d)}{K}] \quad x = \frac{y}{K} \qquad \tau = rt \qquad D = rT_d$$

$$\frac{dx(\tau)}{d\tau} = x(\tau)[1 - x(\tau - D)]$$

Example 2: $D=rT_d = 2.1$ then: Oscillation period $T \approx 4.54T_d$ and $y_{max}/y_{min} \approx 42.3$

Logistic Delay Model : Known Fact V

$$\frac{dy(t)}{dt} = ry(t)[1 - \frac{y(t-T_d)}{K}] \quad x = \frac{y}{K} \quad \tau = rt \quad D = rT_d$$

$$\frac{dx(\tau)}{d\tau} = x(\tau)[1 - x(\tau - D)]$$

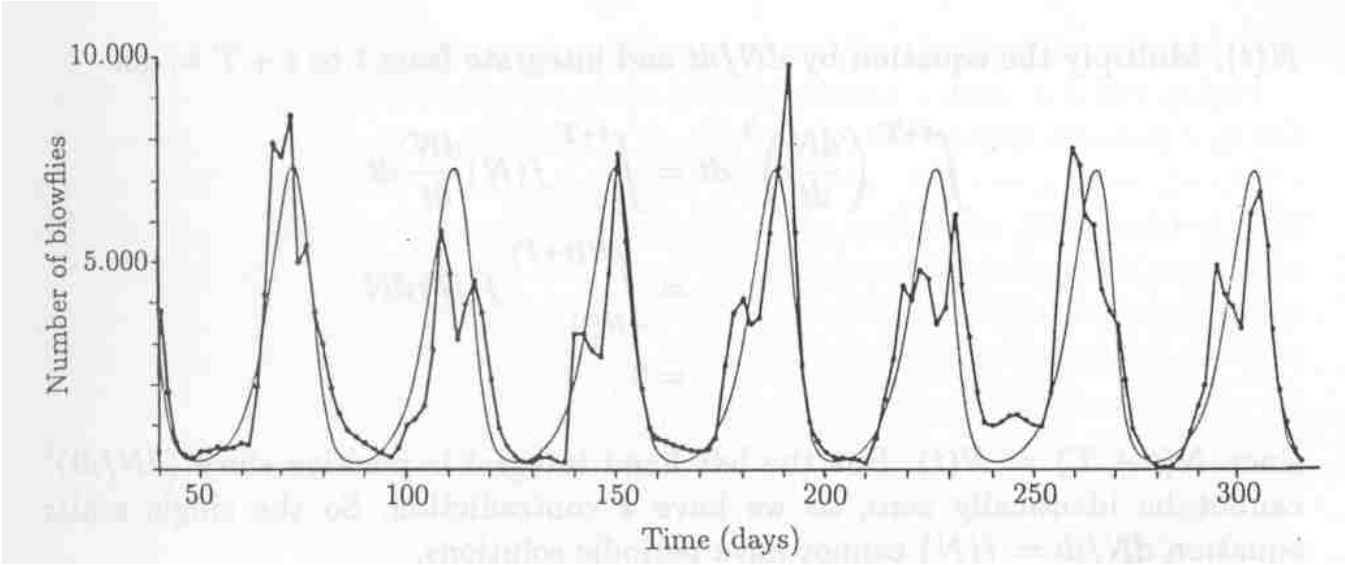
Example 3: $D=rT_d = 2.5$ then: Oscillation period $T \approx 5.36T_d$ and $y_{max}/y_{min} \approx 2930$

The Importance of “Known Facts”

- These can act as “benchmark examples”.
- In the next lecture it will become clear how non-trivial it may be to set correctly the simulation parameters of a dynamic simulator.
- Comparing on-going simulation results to a known benchmark can be very helpful.

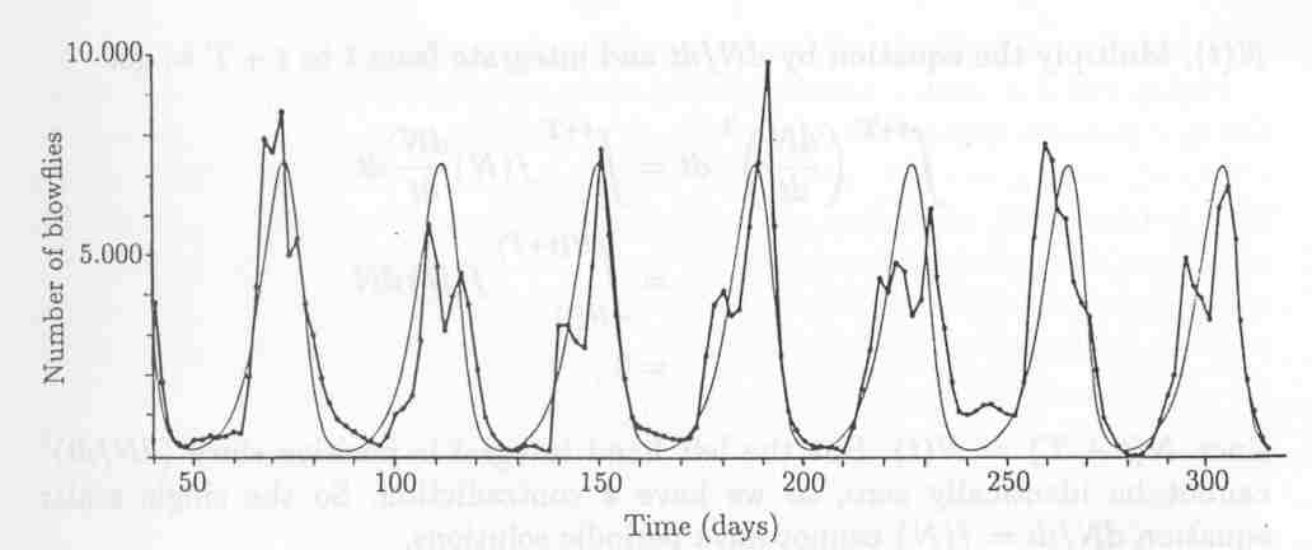
Comparison of Nicholson's (1957) experimental data for the population of the Australian sheep-blowfly and a logistic delay model with $D=rT_d=2.1$

From Murray's book:



- Observed oscillation period was 35-40 days.
- Model was fit using appropriate K, r, T_d values.

Comparison of Nicholson's (1957) experimental data for the population of the Australian sheep-blowfly and a logistic delay model with $D=rT_d=2.1$



- Model implied $T_d \approx 9$ days (using the relationship $T \approx 4.54T_d$. Actual time for larva to mature into adult is around 11 days.
- Oscillation parameters (amplitude, frequency) are actually independent of K (the amount of food available to the insects)

Can a First-order scalar equation <u>without delay</u> have a periodic solution?

Here is a general first-order equation:

$$\frac{dy}{dt} = f(y)$$

Let's see if a periodic solution y(t)=p(t)=p(t+T) works

First-order scalar equation without delay cannot possibly have a periodic solution

Proof:

$$\left(\frac{dy}{dt}\right)^2 = f(y)\frac{dy}{dt} \Rightarrow$$

$$\Rightarrow \int_t^{t+T} \left(\frac{dy}{dt}\right)^2 dt = \int_t^{t+T} f(y)\frac{dy}{dt}\,dt = \int_t^{t+T} f(y)\,dy = 0$$

But the left-hand side is positive. Therefore periodic solution is impossible. The delay plays a crucial role in the periodicity of the solution.