BME 5742 Bio-Systems Modeling and Control

Lecture 11

Linearization Basics

Linearization of the Logistic Model

Why to Linearize? - 1

- Linear model (describing the deviation of the state with respect to an equilibrium) can sometimes help determine if a specific equilibrium state is stable or not.
- In a linear model we can often predict the shape of the time response, using Linear Systems theory. Linearized model can help in the response prediction, if deviation from the equilibrium is not too large.

Why to Linearize? - 2

- One can design a linear controller for the linearized model, using a very rich arsenal of control methods.
- Linear control that is done on a linearized model stays valid as long as the linearized model itself is valid.

How valid is the linearized model?

- The assumption is that state x always stays in the vicinity of the equilibrium x_{eq}.
- How close? This varies from one nonlinear model to another.
- In some models, the linearized model may be valid over a wide range of deviations, in others – over a very small range.
- There are some nonlinear models for which linearization is not valid at all.

How do we determine stability of an equilibrium point? ←→Eigenvalues of the linearized model state transition matrix

- Need to find the <u>eigenvalues</u> λ of matrix A (of the linearized model $dx/dt=Ax+Bu$).
- <u>Eigenvalues:</u> $Ax=\lambda x$, for any vector x.
- Example: If A is 2x2 and x is 2x1, then A has 2 eigenvalues.
- Eigenvalues are found by solving the equation: $\det(\lambda I-A)=0$, where "det" is the determinant and "I" is the identity matrix.

Lyapunov's First Theorem

- Let λ_i i=1,…,N, be the N eigenvalues of the NxN linearized model matrix A.
- Equilibrium x_{eq} is asymptotically stable (i.e. all state-space trajectories near x_{eq} converge to x_{eq}) if all λ_i have negative real part.
- Equilibrium x_{eq} is unstable (i.e. all state-space trajectories near x_{eq} diverge from x_{eq}) if at least one λ_i has positive real part.
- If at least one λ_i has zero real part – Stability cannot be determined from the linearized model. The linearized model may be invalid for all initial conditions for x.

Example: Towards Linearization of the Logistic Model

- Logistic model: $dy/dt=ay(1-y/K)$.
- As we know, the two equilibria are: $y_{e1}=0$ and $y_{e2}=K$.
- Let's consider a variable x, where $x=y-y_e$:

  $dy/dt=d(x+y_e)/dt=a(x+y_e)(1-(x+y_e)/K)$
- Note that $dy_e/dt=ay_e(1-y_e/K)$ which cancels out of the equation above.

Linearization of Logistic Model: Nonlinear model for the deviations

- (Repeating from the previous page) $x=y-y_e$:
  $dy/dt=d(x+y_e)/dt=a(x+y_e)(1-(x+y_e)/K)$
- Note that $dy_e/dt=ay_e(1-y_e/K)$ which cancels out of the equation. What is left?
  $dx/dt=ax-ax^2/K-2axy_e/K$
- This is an accurate nonlinear model for x, around any system point $y=y_e$, even if it is not an equilibrium.

Linearization of the Logistic Model: Stability of $y_e=0$

- (Repeating from the previous page) $dx/dt=ax-ax^2/K-2axy_e/K$
- Let us now assume that x is "small", and neglect terms such as x^2 (with respect to x):
- Case 1: Let $y_e=y_{e1}=0$. Then: $dx/dt\approx ax$. Here $A=a=\lambda>0$ as $a>0$ → Exponentially increasing solution. Therefore $y_e=0$ is unstable.

Linearization of the Logistic Model: Stability of y_e=K

- $dx/dt=ax-ax^2/K-2axy_e/K$
- Let us now assume that x is "small", and neglect all terms such as x^2:
- Case 2: Let $y_e=y_{e2}=K$. Then:

  $dx/dt\approx -ax$. Here $A= -a=\lambda<0$ as $a>0$. Therefore y_e=K is stable.

Linear Systems

$$\frac{dx_1}{dt} = a_{11}x_1 + a_{12}x_2$$

$$\frac{dx_2}{dt} = a_{21}x_1 + a_{22}x_2$$

- The above is the most general autonomous (i.e. no external input signals) second order linear system
- The a coefficients are constant.

Equilibrium of Linear Systems

$$\frac{dx_1}{dt} = 0 = a_{11}x_1 + a_{12}x_2$$

$$\frac{dx_2}{dt} = 0 = a_{21}x_1 + a_{22}x_2$$

- Linear systems have only one equilibrium point: $x_{1e}=0$ and $x_{2e}=0$
- The equilibrium state (0,0) can be of several types, depending on the a coefficients

Linear System Equilibrium Point Types

$$\det(\lambda I - A) = 0$$

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$$

$$\det(\lambda I - A) = \det\begin{pmatrix} \lambda - a_{11} & -a_{12} \\ -a_{21} & \lambda - a_{22} \end{pmatrix} = 0$$

$$(\lambda - a_{11})(\lambda - a_{22}) - a_{12}a_{21} = 0 = \lambda^2 - (a_{11} + a_{22})\lambda + a_{11}a_{22} - a_{12}a_{21}$$

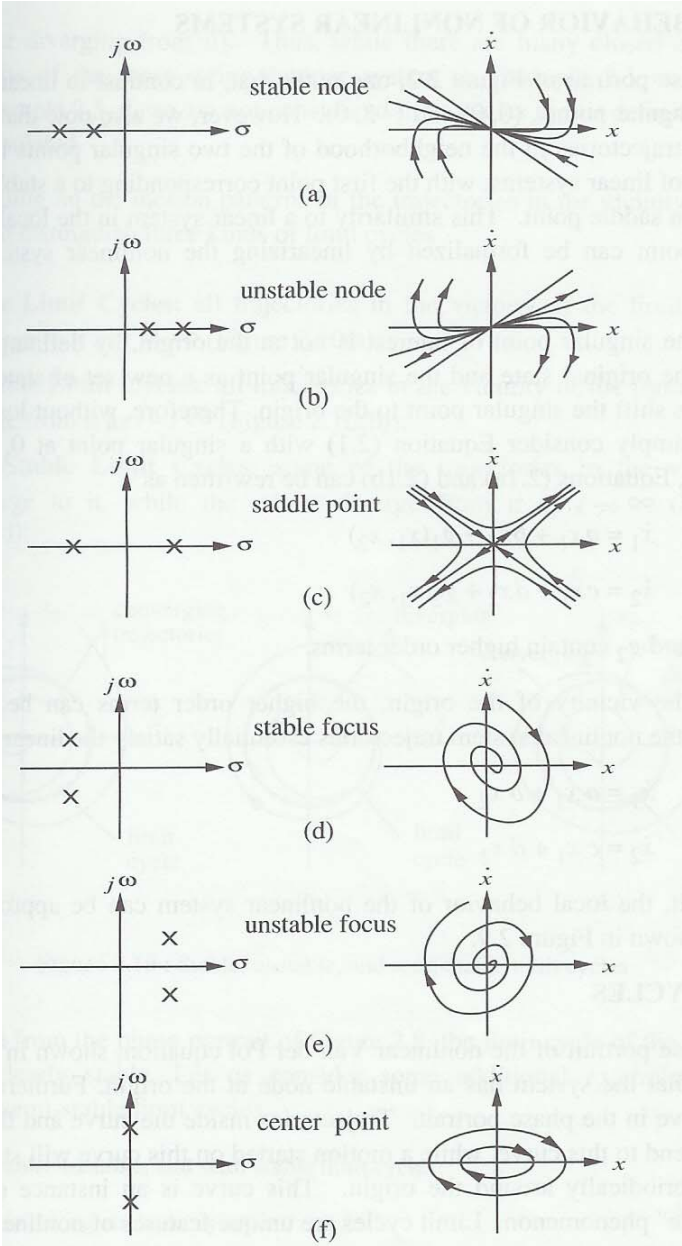

Figure 2.9 : Phase-portraits of linear systems

Equilibrium Types

- Node if eigenvalues are both real and have the same sign
- Focus if eigenvalues are complex
- Saddle if eigenvalues are both real but have opposing signs
- Center if eigenvalues are purely imaginary