BME 5742 Bio-Systems Modeling and Control

Lecture 10

Competition and Symbiosis Models

Competition Models

- Two or more species compete for the same limited food resource, or in some way inhibit each others growth.
- Example: Competition may be for territory which is directly related to food resources.
- General Observation ("Principle of Competitive Exclusion"): When two species compete, one usually becomes extinct.

Simple Competition Model based on two logistic-growth populations:

$$\frac{dN_1}{dt} = r_1 N_1 (1 - \frac{N_1}{K_1} - b_{12} \frac{N_2}{K_1})$$

$$\frac{dN_2}{dt} = r_2 N_2 (1 - \frac{N_2}{K_2} - b_{21} \frac{N_1}{K_2})$$

All constants are positive.

Birth Rates: r_1, r_2

Carrying Capacities: K_1, K_2

Competition Model : Modeling the competitive effect

$$\frac{dN_1}{dt} = r_1 N_1 (1 - \frac{N_1}{K_1} - b_{12} \frac{N_2}{K_1})$$

$$\frac{dN_2}{dt} = r_2 N_2 (1 - \frac{N_2}{K_2} - b_{21} \frac{N_1}{K_2})$$

The competitive interaction is proportional to $N_1 N_2$.

The interaction causes both growth rates to <u>decrease</u>, not necessarily by same amount.

Competition Model Normalization

- Details are not important.
- We can reduce the number of model parameters from 6 to 3.

$$u_1 = \frac{N_1}{K_1} \qquad u_2 = \frac{N_2}{K_2} \qquad \tau = r_1 t$$

Normalized Competition Model Parameters

$$\rho = \frac{r_2}{r_1} \qquad a_{12} = b_{12}\frac{K_2}{K_1} \qquad a_{21} = b_{21}\frac{K_1}{K_2}$$

Normalized Competition Model

$$\rho = \frac{r_2}{r_1} \qquad a_{12} = b_{12}\frac{K_2}{K_1} \qquad a_{21} = b_{21}\frac{K_1}{K_2}$$

$$u_1 = \frac{N_1}{K_1} \qquad u_2 = \frac{N_2}{K_2} \qquad \tau = r_1 t$$

$$\frac{du_1}{d\tau} = u_1(1 - u_1 - a_{12}u_2)$$

$$\frac{du_2}{d\tau} = \rho u_2(1 - u_2 - a_{21}u_1)$$

Competition Model Equilibria – equilibria locations don't depend on ρ, but equilibria stability does

$$\frac{du_1}{d\tau} = 0 = u_1(1 - u_1 - a_{12}u_2)$$

$$\frac{du_2}{d\tau} = 0 = \rho u_2(1 - u_2 - a_{21}u_1)$$

4 equilibria: (Stability properties – without proof. More will be said in the respective simulation tutorial)

(0,0) is an unstable equilibrium.

(1,0) is stable if $a_{21}>1$, and unstable if $a_{21}<1$.

Competition Model Equilibria 2

$$\frac{du_1}{d\tau} = 0 = u_1(1 - u_1 - a_{12}u_2)$$

$$\frac{du_2}{d\tau} = 0 = \rho u_2(1 - u_2 - a_{21}u_1)$$

(0,1) is stable if $a_{12}>1$.

$((1-a_{12})/(1-a_{12}a_{21}),(1-a_{21})/(1-a_{12}a_{21}))$ exists only if both parameters are simultaneously <1, or simultaneously >1.

Competition Model Equilibria 3

$$\frac{du_1}{d\tau} = 0 = u_1(1 - u_1 - a_{12}u_2)$$

$$\frac{du_2}{d\tau} = 0 = \rho u_2(1 - u_2 - a_{21}u_1)$$

Depending on ρ, $((1-a_{12})/(1-a_{12}a_{21}),(1-a_{21})/(1-a_{12}a_{21}))$ is sometimes stable if both "a parameters" are simultaneously <1 ("non-aggressive competition").

Here both (1,0) and (0,1) are unstable.

Competition Model Equilibria 4

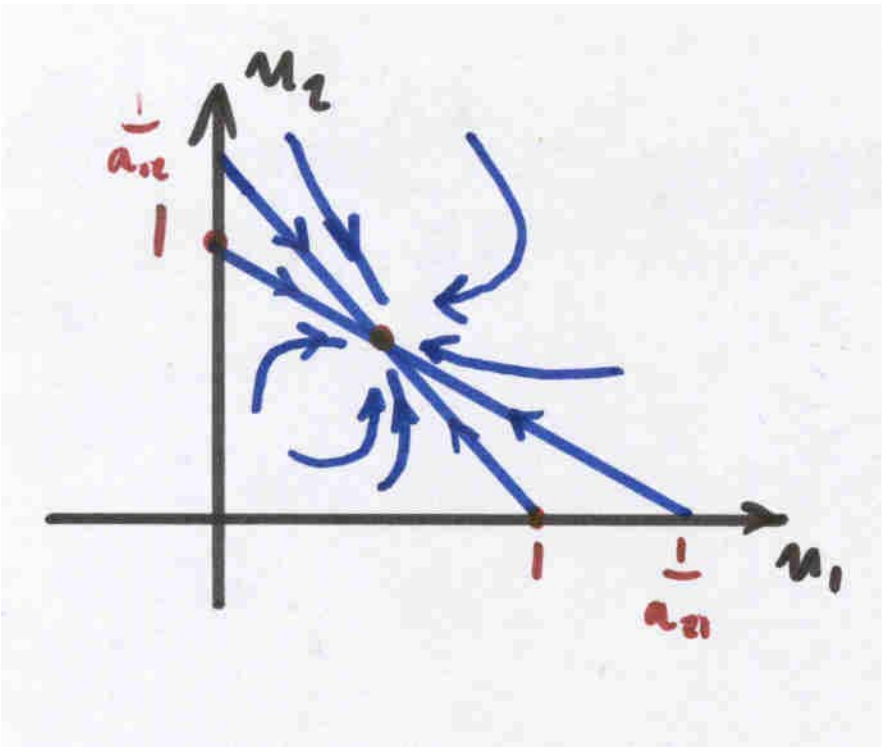
$$\frac{du_1}{d\tau} = 0 = u_1(1 - u_1 - a_{12}u_2)$$

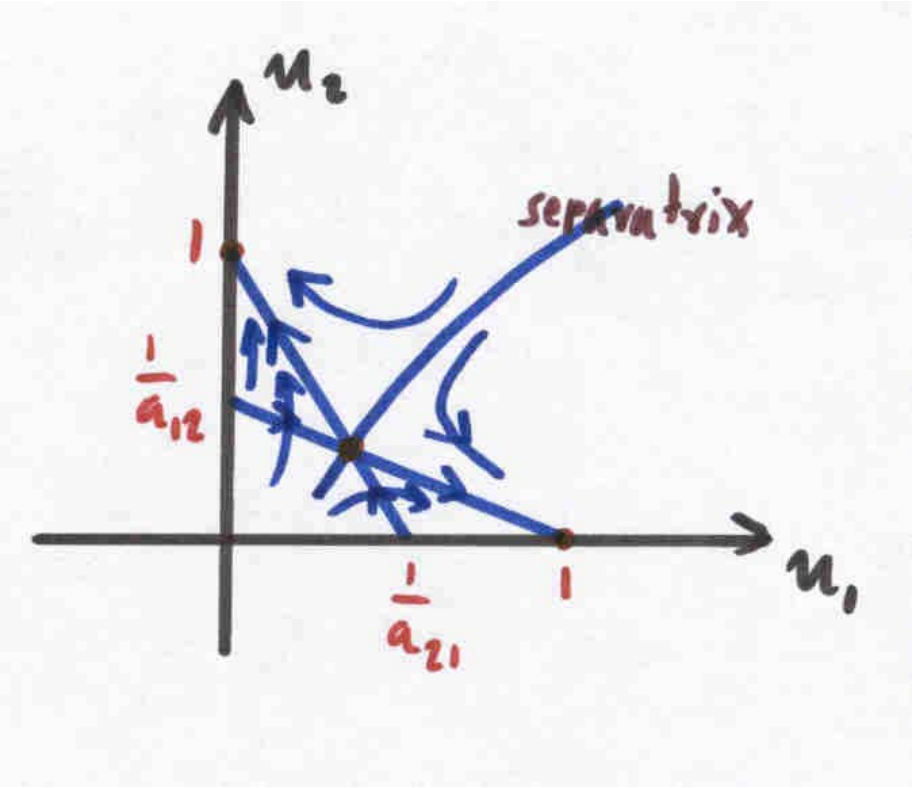
$$\frac{du_2}{d\tau} = 0 = \rho u_2(1 - u_2 - a_{21}u_1)$$

$((1-a_{12})/(1-a_{12}a_{21}),(1-a_{21})/(1-a_{12}a_{21}))$ is unstable if both parameters a_{12} and a_{21} are simultaneously >1. Here both (1,0) and (0,1) are stable, each has its own domain of attraction (i.e. set of initial states from which solution converge to given equilibrium).

Competition Model Solution Cases

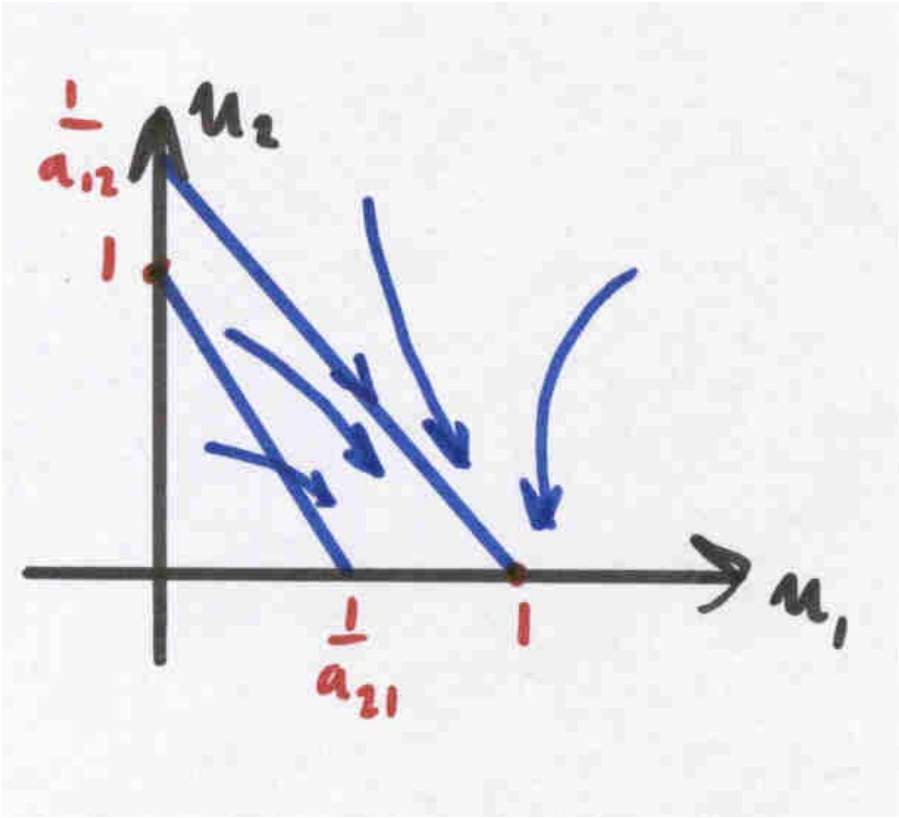
[Based on Murray's book]

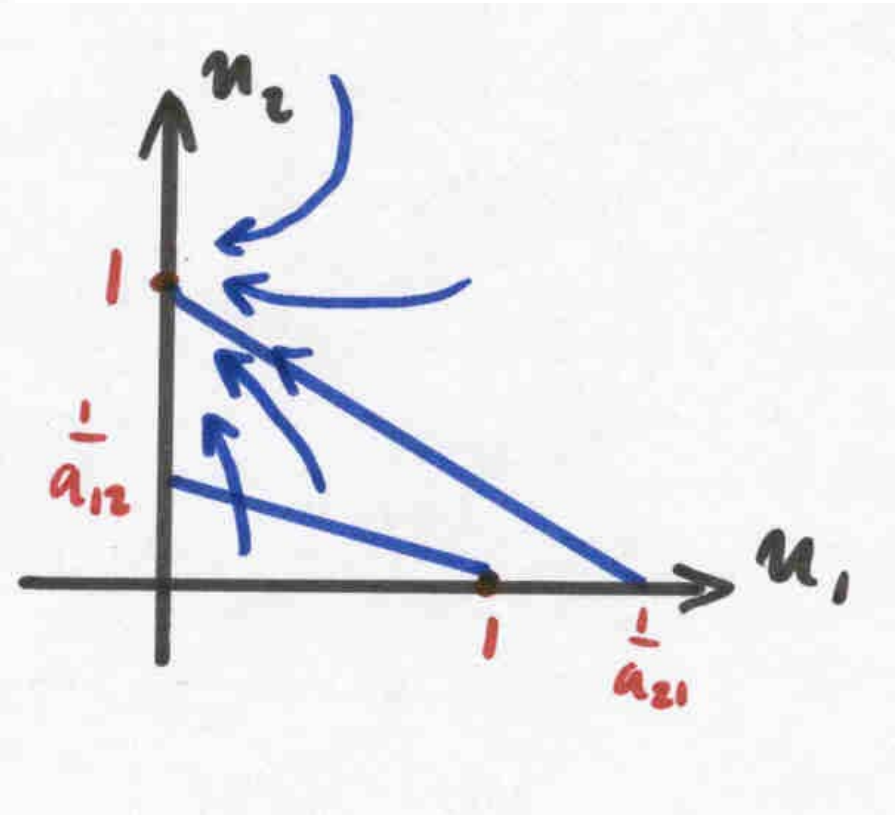




Competition Model Solution Cases

[Based on Murray's book]





Simple Mutualism (Symbiosis) Model based on Lotka-Volterra:

$$\frac{dN_1}{dt} = r_1 N_1 (1 - \frac{N_1}{K_1} + b_{12} \frac{N_2}{K_1})$$

$$\frac{dN_2}{dt} = r_2 N_2 (1 - \frac{N_2}{K_2} + b_{21} \frac{N_1}{K_2})$$

All constants are positive.

Birth Rates: r_1, r_2

Carrying Capacities: K_1, K_2

Symbiosis Model : Modeling the mutualism effect

$$\frac{dN_1}{dt} = r_1 N_1 (1 - \frac{N_1}{K_1} + b_{12} \frac{N_2}{K_1})$$

$$\frac{dN_2}{dt} = r_2 N_2 (1 - \frac{N_2}{K_2} + b_{21} \frac{N_1}{K_2})$$

The beneficial mutual interaction is proportional to $N_1 N_2$.

The interaction causes both growth rates to <u>increase</u>, not necessarily by same amount.

Symbiosis Model Normalization

- It is done in exactly the same way as the Competition model.
- We can reduce the number of model parameters from 6 to 3.

$$u_1 = \frac{N_1}{K_1} \qquad u_2 = \frac{N_2}{K_2} \qquad \tau = r_1 t$$

Normalized Symbiosis Model Parameters

$$\rho = \frac{r_2}{r_1} \qquad a_{12} = b_{12}\frac{K_2}{K_1} \qquad a_{21} = b_{21}\frac{K_1}{K_2}$$

Normalized Symbiosis Model

$$\rho = \frac{r_2}{r_1} \qquad a_{12} = b_{12}\frac{K_2}{K_1} \qquad a_{21} = b_{21}\frac{K_1}{K_2}$$

$$u_1 = \frac{N_1}{K_1} \qquad u_2 = \frac{N_2}{K_2} \qquad \tau = r_1 t$$

$$\frac{du_1}{d\tau} = u_1(1 - u_1 + a_{12}u_2)$$

$$\frac{du_2}{d\tau} = \rho u_2(1 - u_2 + a_{21}u_1)$$

Symbiosis Model Equilibria 1

$$\frac{du_1}{d\tau} = 0 = u_1(1 - u_1 + a_{12}u_2)$$

$$\frac{du_2}{d\tau} = 0 = \rho u_2(1 - u_2 + a_{21}u_1)$$

4 equilibria: (Stability properties – without proof)

(0,0),(1,0) and (0,1) are all unstable equilibria.

Symbiosis Model Equilibria 2

$$\frac{du_1}{d\tau} = 0 = u_1(1 - u_1 + a_{12}u_2)$$

$$\frac{du_2}{d\tau} = 0 = \rho u_2(1 - u_2 + a_{21}u_1)$$

$((1+a_{12})/(1-a_{12}a_{21}),(1+a_{21})/(1-a_{12}a_{21}))$
exists only if $1-a_{21}a_{12}>0$.

Symbiosis Model Equilibria 3

$$\frac{du_1}{d\tau} = 0 = u_1(1 - u_1 + a_{12}u_2)$$

$$\frac{du_2}{d\tau} = 0 = \rho u_2(1 - u_2 + a_{21}u_1)$$

If $1-a_{12}a_{21}<0$, there are only three unstable equilibrium points. All trajectories go to infinity.

It's an indication that model is unrealistic.

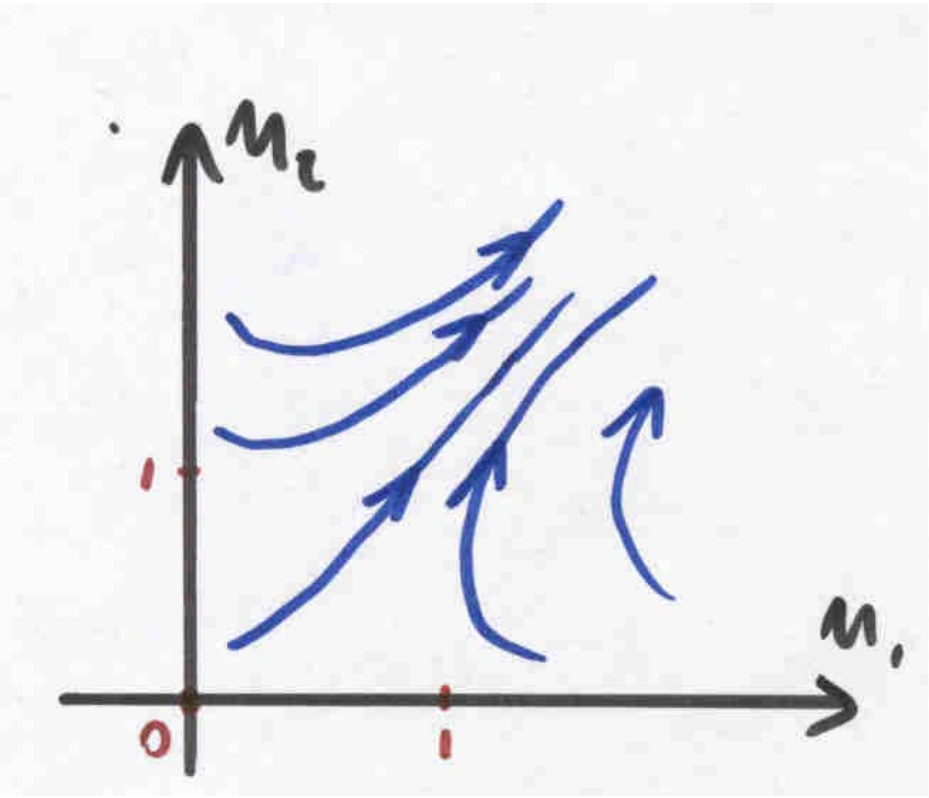
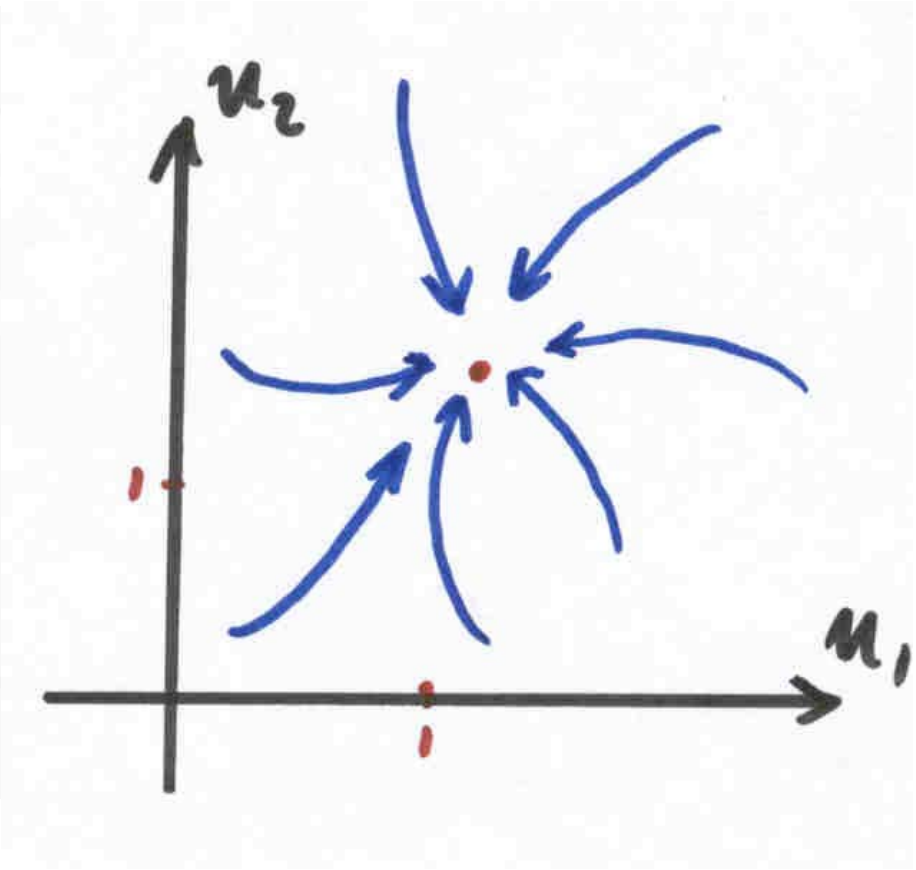
Symbiosis Model Equilibria 4

$$\frac{du_1}{d\tau} = 0 = u_1(1 - u_1 + a_{12}u_2)$$

$$\frac{du_2}{d\tau} = 0 = \rho u_2(1 - u_2 + a_{21}u_1)$$

If $1-a_{12}a_{21}>0$ then all trajectories go to the only stable equilibrium point $((1+a_{12})/(1-a_{12}a_{21}),(1+a_{21})/(1-a_{12}a_{21}))$.

Note that the coordinates of this equilibrium are each >1! It's an "improvement" compared to the case $a_{12}=a_{21}=0$, where each population goes to a normalized value of 1.

Symbiosis Model Solution Cases

[Based on Murray's book]