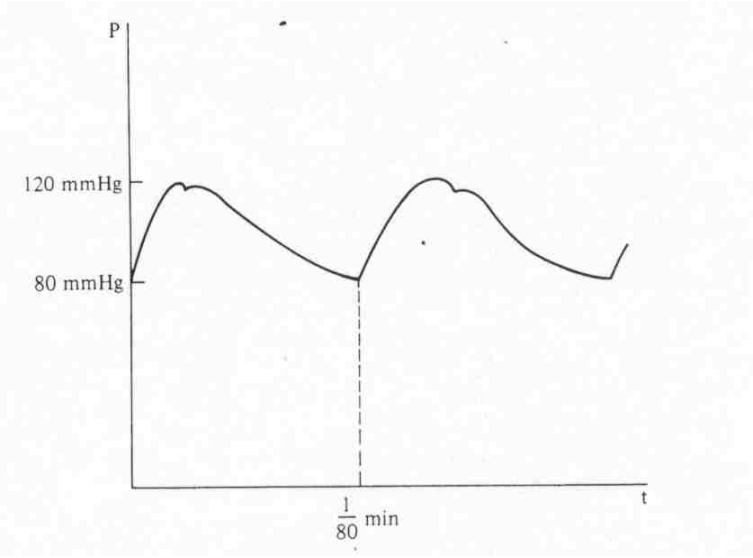
BME 5742 Bio-Systems Modeling and Control

Lecture 44
Heart & Blood Circulation
Dynamic Equations

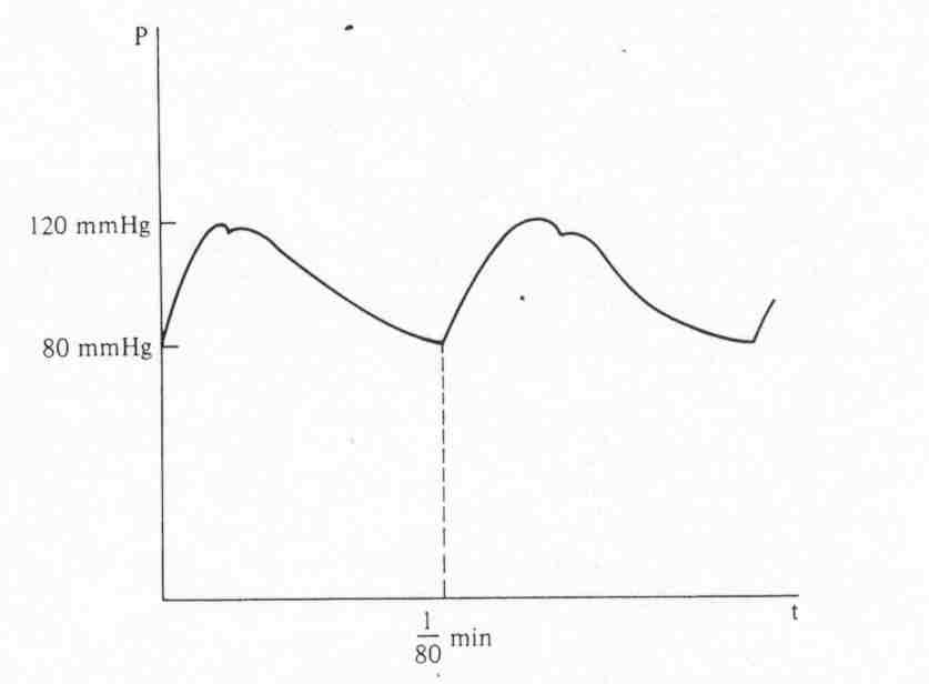
Dynamic vs. Static Models



- So far we looked at "static" Flow – Pressure – Volume models, calling it steady-state flow models. These were <u>algebraic equations</u>.
- Variables are, as we all know, time-varying (almost periodic). See typical $P_{sa}(t)$ above.
- "Dynamic Models" ←→ <u>Differential Equations</u>

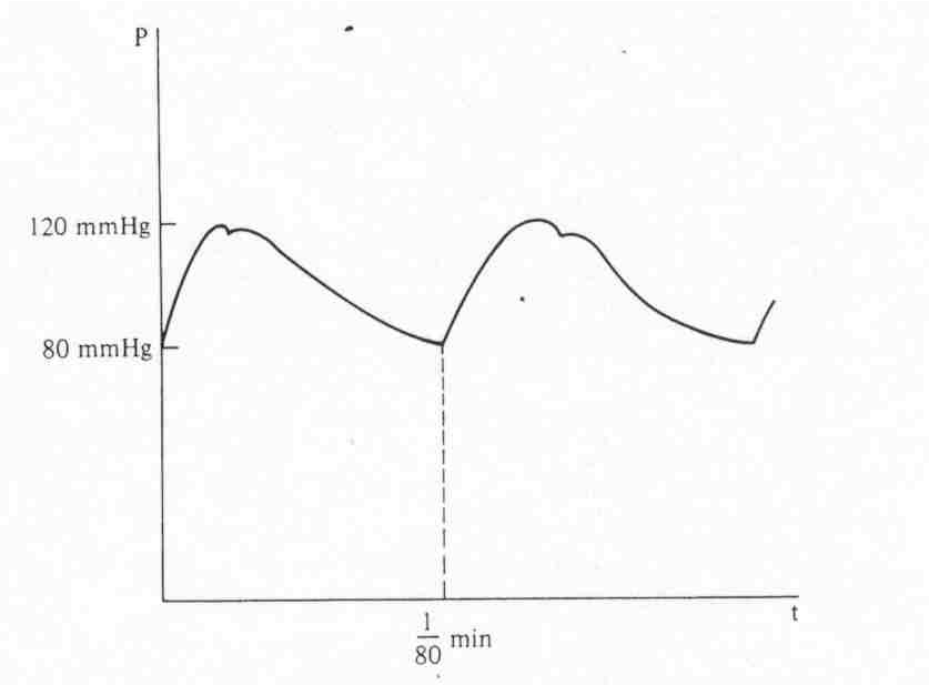
Systemic Arterial Pressure Wave

- Maximum (systolic) pressure is typically 120 mmHg.
- Minimum (diastolic) pressure is typically 80 mmHg.
- Period is typically 1/80 minutes.



Systemic Arterial Pressure Explained Qualitatively

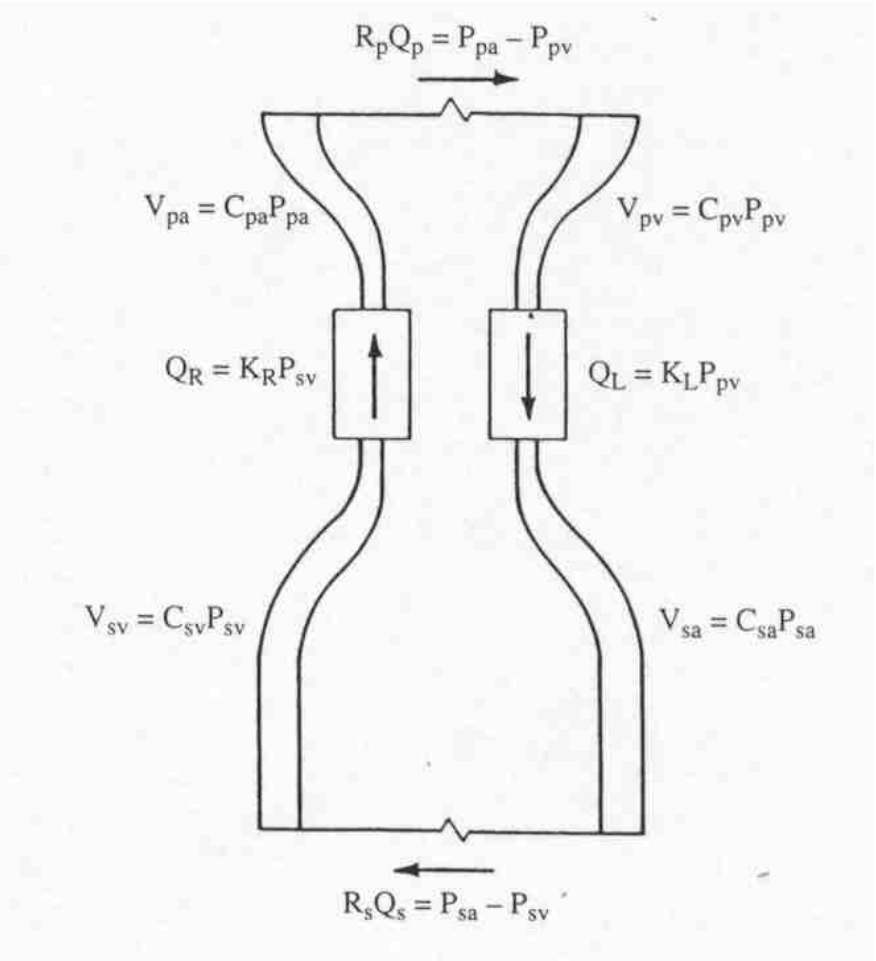
- During contractions of the ventricles, blood pressure rises rapidly.
- During ventricles relaxation, out-flowing valves are closed, there is zero input flow into the arteries.
- Blood pressure during diastole falls, almost exponentially.



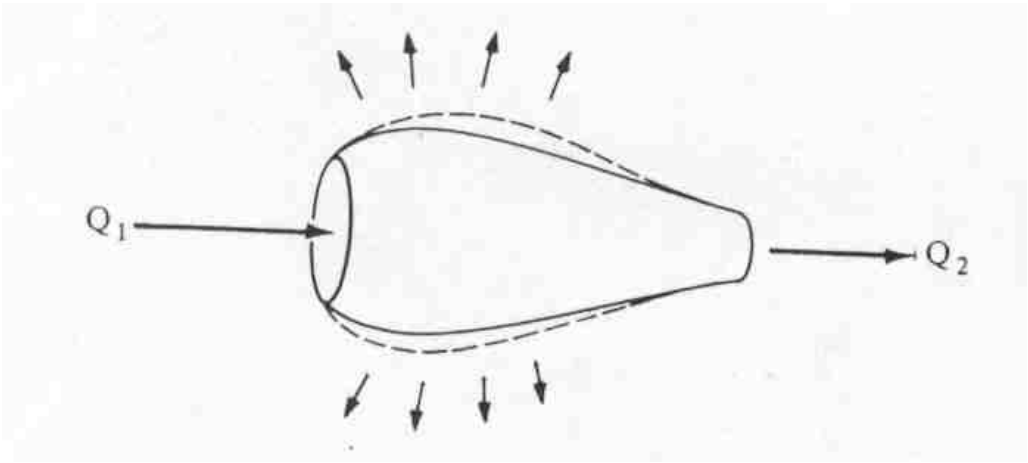
Question: How do these levels depend on the system parameters?

Simulation Model – Preliminary Ideas

The 8-component diagram, discussed before, can form a basis for a dynamic simulation ("flow circuit" with 4 capacitors, 2 resistors, and two synchronized and periodic current sources).

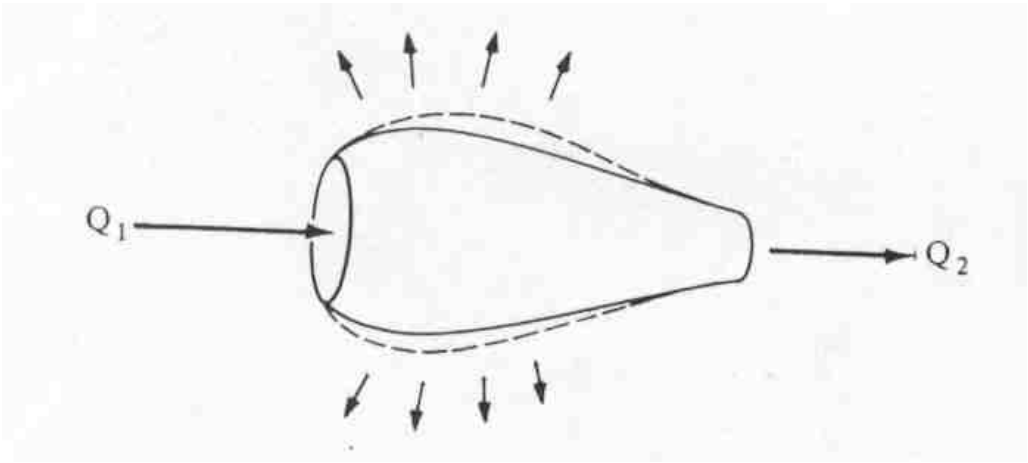


Compliance Vessels that are not in steady-state



- In general, the inflow and outflow may not be equal: $Q_1(t) \neq Q_2(t)$.
- Whenever there is no inflow-outflow equality, the volume of the vessel is changing: $dV(t)/dt = Q_1(t)-Q_2(t)$. At steady-state, when V=const, we have $Q_1(t)=Q_2(t)$.

Basic Pressure-Flow Differential Equations in a Compliant vessel



- We assume that the compliance of blood vessels is more or less constant (contrary to the heart ventricles, in which the compliance is a periodic time function).
- $V(t)=CP(t) \rightarrow \boxed{CdP(t)/dt = Q_1(t)-Q_2(t)}$

Systemic Arterial Pressure Equation During Systole

- Regarding the system of body arteries:
- $C=C_{sa}$ (systemic arterial compliance)
- $P=P_{sa}$ (systemic arterial pressure)
- $Q_1(t)=Q_L(t)$ (left heart cardiac output)
- $Q_2(t)=Q_s(t)$ (blood flow through the systemic tissues.

$$C\frac{dP(t)}{dt}=Q_1(t)-Q_2(t)\Leftrightarrow$$

$$\Leftrightarrow C_{sa}\frac{dP_{sa}(t)}{dt}=Q_L(t)-Q_s(t)$$

Systemic Arterial Pressure Equation During Systole taking into account the systemic resistance

- We assume, for the moment, a constant systemic resistance R_s.
- Systemic arterial differential equation is coupled (via $P_{sv}(t)$) to the systemic venous differential equation.
- As an approximation we can neglect the above coupling.

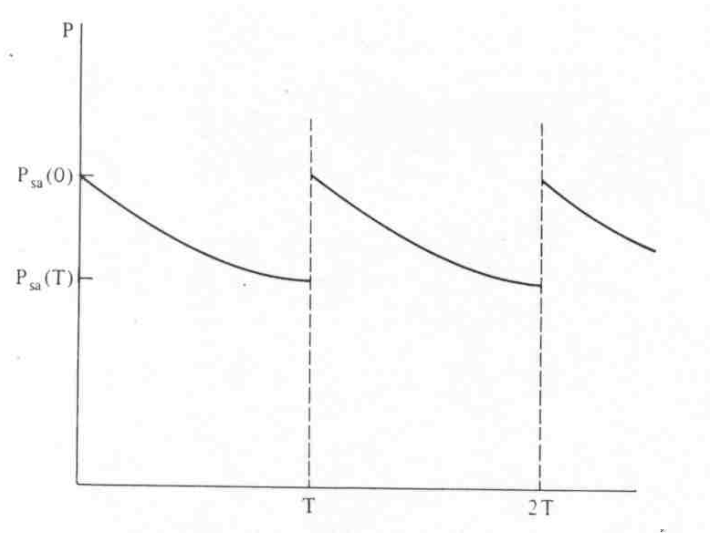
$$C_{sa}\frac{dP_{sa}(t)}{dt}(t) = Q_L(t) - Q_s(t)$$

$$Q_s(t) = \frac{P_{sa}(t) - P_{sv}(t)}{R_s}$$

$$P_{sv}(t) << P_{sa}(t) \Rightarrow$$

$$\Rightarrow C_{sa}\frac{dP_{sa}(t)}{dt}(t) \approx Q_L(t) - \frac{P_{sa}(t)}{R_s}$$

Systemic Arterial Pressure During Diastole: Exponential pressure decay



In this figure we neglect the finite pressure rise time during systole

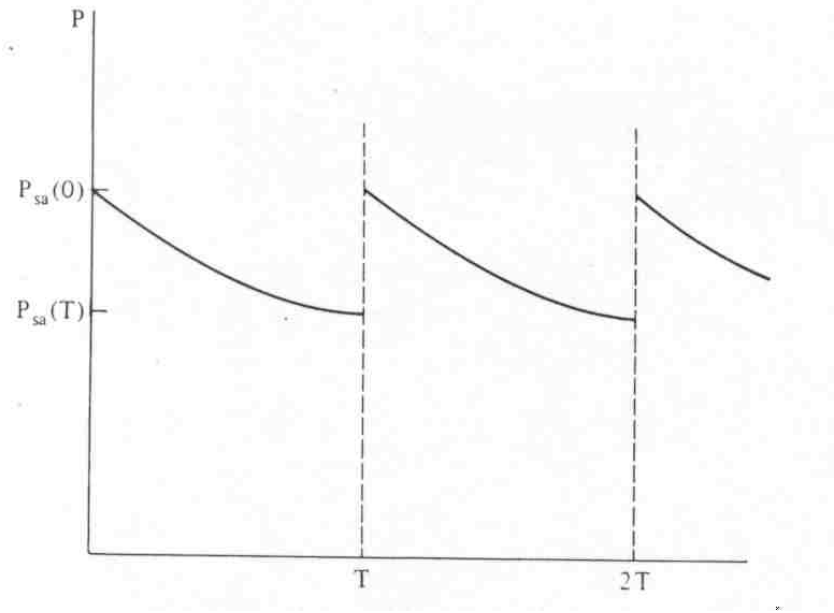
$$Q_L(t) = 0 \Rightarrow$$

$$\Rightarrow C_{sa}\frac{dP_{sa}(t)}{dt}(t) = Q_L(t) - \frac{P_{sa}(t)}{R_s} \Rightarrow$$

$$\Rightarrow \boxed{P_{sa}(t) = P_{sa}(0)e^{-\frac{t}{R_s C_{sa}}}}$$

$P_{sa}(0)$ is not yet determined

Systolic ($P_{sa}(0)$) and Diastolic ($P_{sa}(T)$) Pressures depend on the ejected stroke volume ΔV_0:



As we neglect the pressure rise time during systole, we essentially assume that ΔV_0 is all ejected at once

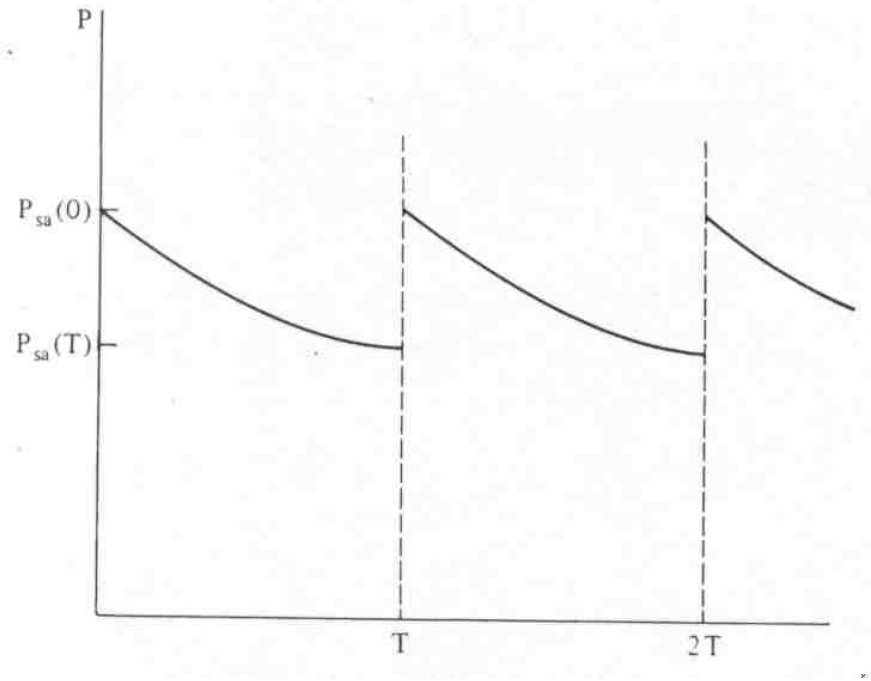
$V_{sa}=C_{sa}P_{sa}+V_d$ → →

$\Delta V_0=C_{sa}\Delta P_{sa}$

Where $\Delta P_{sa}=P_{sa}(0)-P_{sa}(T)$

$$P_{sa}(t) = P_{sa}(0)e^{-\frac{t}{R_s C_{sa}}}$$

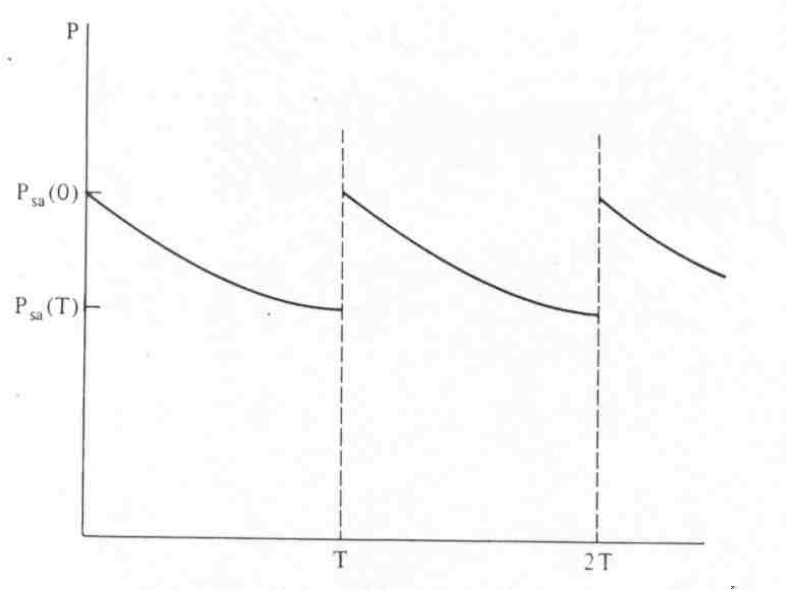
We solve for the Systolic ($P_{sa}(0)$) and Diastolic ($P_{sa}(T)$) Pressures using the following two equations with two unknowns:



$$P_{sa}(T) = P_{sa}(0)e^{-\frac{T}{R_s C_{sa}}} = \Theta P_{sa}(0)$$

$$\frac{\Delta V_0}{C_{sa}} = P_{sa}(0) - P_{sa}(T)$$

Systolic ($P_{sa}(0)$) and Diastolic ($P_{sa}(T)$) Pressure Solutions



$$P_{sa}(T) = P_{sa}(0)\boxed{e^{-\frac{T}{R_s C_{sa}}} = \Theta}P_{sa}(0)$$

$$\frac{\Delta V_0}{C_{sa}} = P_{sa}(0) - P_{sa}(T)$$

$$\boxed{\begin{aligned} P_{sa}(0) &= \frac{\Delta V_0}{C_{sa}(1-\Theta)} \\ P_{sa}(T) &= \frac{\Theta \Delta V_0}{C_{sa}(1-\Theta)} \end{aligned}}$$

Causes of Systolic ($P_{sa}(0)$) and Diastolic ($P_{sa}(T)$) Pressure Changes (suggested computational scenarios)

- Take a person with 120/80 BP and introduce % changes in the various parameters. See the effect.
- Is it possible for the systolic pressure to go up whereas diastolic pressure goes down (say 150/70) – under what circumstances?

$$e^{-\frac{T}{R_s C_{sa}}} = \Theta$$

$$P_{sa}(0) = \frac{\Delta V_0}{C_{sa}(1-\Theta)}$$

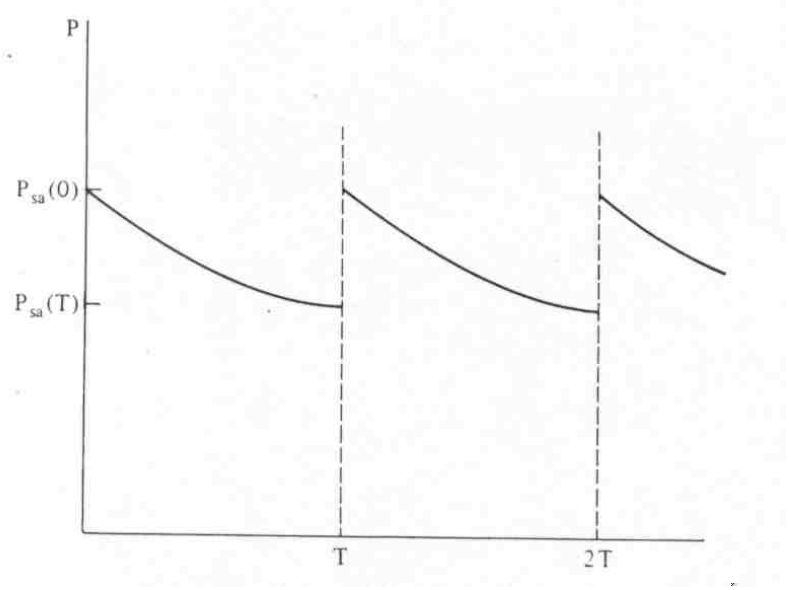
$$P_{sa}(T) = \frac{\Theta \Delta V_0}{C_{sa}(1-\Theta)}$$

The Mean Arterial Pressure

$$e^{-\frac{T}{R_s C_{sa}}} = \Theta$$

$$P_{sa}(0) = \frac{\Delta V_0}{C_{sa}(1-\Theta)}$$

$$P_{sa}(T) = \frac{\Theta \Delta V_0}{C_{sa}(1-\Theta)}$$



Exponential Waveform

$$\boxed{\langle P_{sa} \rangle} = \frac{1}{T}\int_0^T P_{sa}(t)dt = \ldots = \boxed{\frac{R_s \Delta V_0}{T}}$$

Interpretation of the Mean Arterial Pressure

$$\langle P_{sa} \rangle = \frac{R_s \Delta V_0}{T} = R_s Q$$

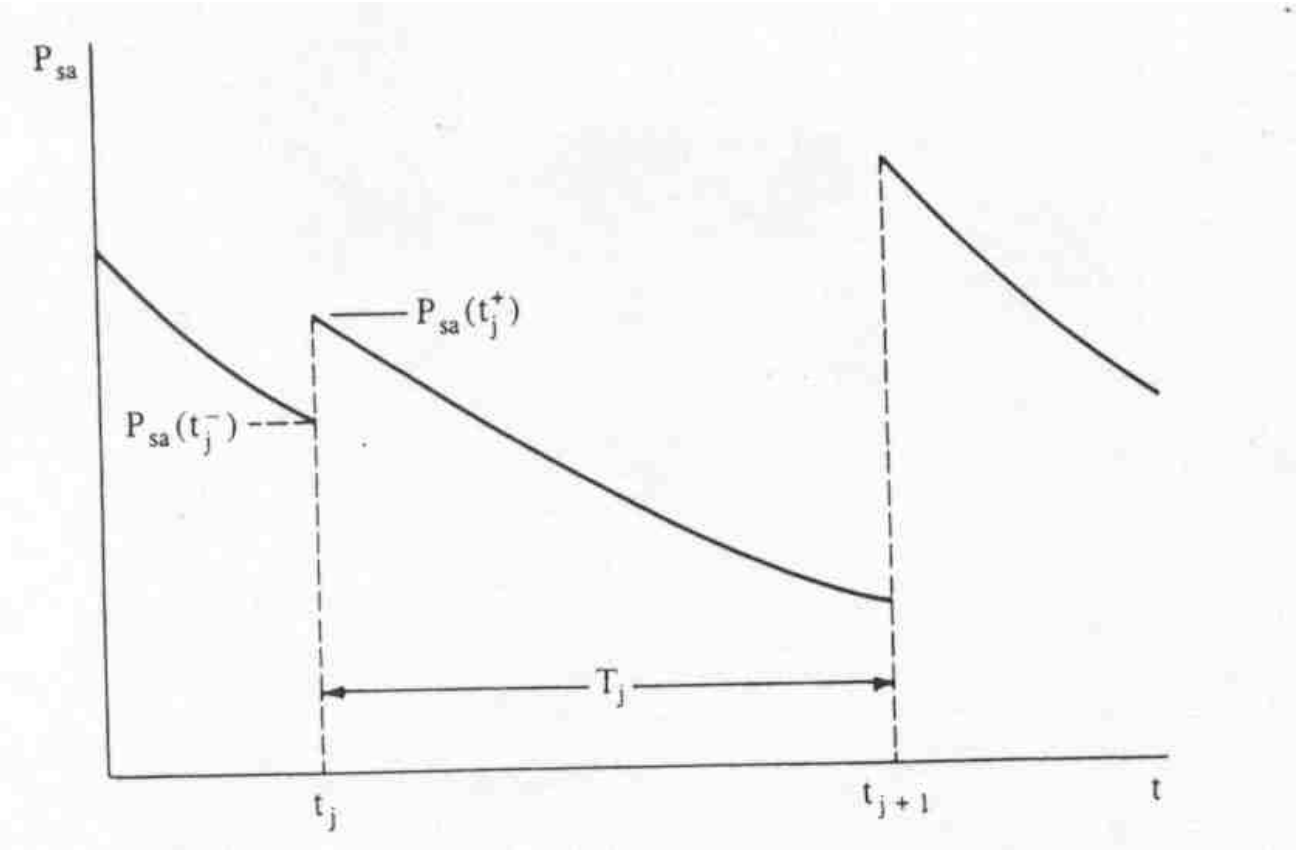
- We note that $\Delta V_0/T$ is exactly the cardiac output Q.
- In our previous steady-state analysis we used $P_{sa}-P_{sv}=R_sQ$, so we see that such steady-state analysis can be relevant (very approximately) to pulseatile circulation, using variables means.

Solution is not always periodic

- We solved the pressure dynamic equations, assuming a perfectly periodic solution.
- The dynamic model can show transient effects due to sudden changes. Eventually the periodic steady-state resumes.
- Extreme case example: Heart revival after a period of cardiac arrest. $P_{sa}(0)$ may be very low, and it builds up over the next few heart beats.

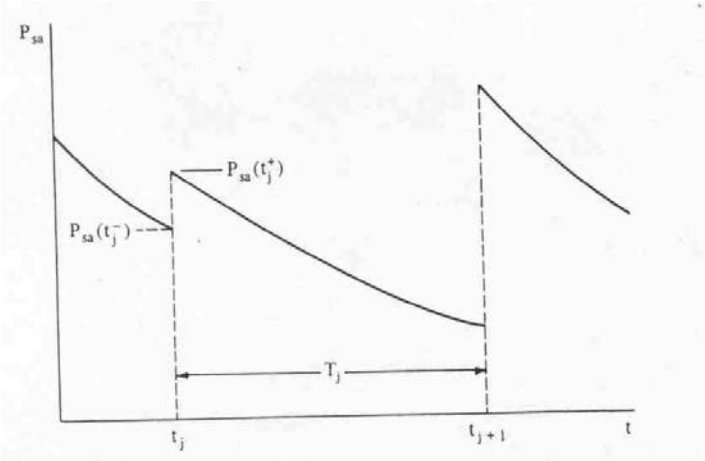
Transient Analysis Solution Method

We assume irregular heart beats, where both the stroke volume and the rhythm may vary from beat to beat.



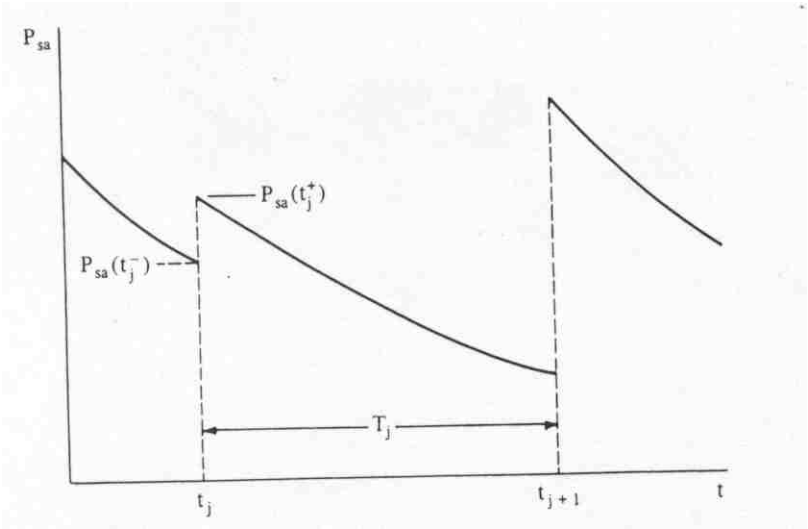
Beats counting index is $j = 1, 2, \ldots$

Transient Analysis: Irregular Heartbeats Notation



- t_j= The time that jth heart beat occurs.
- $P_{sa}(t_j^-)$= diastolic arterial pressure just before ejection.
- $P_{sa}(t_j^+)$= systolic arterial pressure just after ejection.

Diastolic Pressure Equation and Solution -1



$$C_{sa}\frac{dP_{sa}(t)}{dt}(t) = -\frac{P_{sa}(t)}{R_s} \Rightarrow$$

$$\Rightarrow P_{sa}(t) = P_{sa}(t_j^{\ +})e^{-\frac{t-t_j}{R_s C_{sa}}} \qquad t_j \le t < t_{j+1}$$

Diastolic Pressure just before Beat j+1:

$$C_{sa}\frac{dP_{sa}(t)}{dt}(t) = -\frac{P_{sa}(t)}{R_s} \Rightarrow$$

$$\Rightarrow P_{sa}(t) = P_{sa}(t_j^{\,+})e^{-\frac{t-t_j}{R_sC_{sa}}} \qquad t_j \le t < t_{j+1}$$

$$P_{sa}(t_{j+1}^{\,-}) = P_{sa}(t_j^{\,+})e^{-\frac{t_{j+1}^{\,-}-t_j}{R_sC_{sa}}} = P_{sa}(t_j^{\,+})e^{-\frac{T_j}{R_sC_{sa}}}$$

$$\boxed{\begin{aligned} &P_{sa}(t_{j+1}^{\,-}) = P_{sa}(t_j^{\,+})\Theta_j \\ &\Theta_j = e^{-\frac{T_j}{R_sC_{sa}}} \\ &T_j = t_{j+1}^{\,-} - t_j = t_{j+1} - t_j \end{aligned}}$$

Jump in arterial pressure at $t=t_j$

$$P_{sa}(t_j^{+}) = P_{sa}(t_j^{-}) + \frac{\Delta V_j}{C_{sa}}$$

So, if a sequence of times t_j and stroke volumes ΔV_j are given together with the constant values of C_{sa} and R_s, and if we know the arterial pressure just before the first beat, then we can find all subsequent systolic and diastolic pressures, no matter how irregular the beats.