BME 5742 Bio-Systems Modeling and Control

Lecture 3

Malthusian and Logistic Models

References for this lecture

- J.D. Murray “Mathematical Biology”, Second corrected Edition, Springer 1993
- Edward K. Yeargers, Ronald W. Shonkwiler & James V. Herod “An Introduction to the Mathematics of Biology” (With Computer Algebra Models), Birkhauser 1996.

Thomas Robert Malthus (1766-1834)

- British cleric and scholar, influential in the fields of political economy and demography.
- published the "Essay on the Principle of Population," which argued that population multiplies geometrically and food arithmetically. Therefore, the population will outstrip the food supply.

Population Per-Capita Growth Rate

$$r = \frac{(birth_rate) - (death_rate)}{population_size}$$

If r = const > 0 → Population increases exponentially

If r = const < 0 → Population decays exponentially

Malthus Model (1798)

Model assumption: Rate of change of a population is proportional to the population size

Let

y = population size

and

y_0 = initial population size at t = 0.

$$\frac{dy}{dt} = ry$$

Mathematical definition of Population Per-Capita Growth Rate using Malthus Model

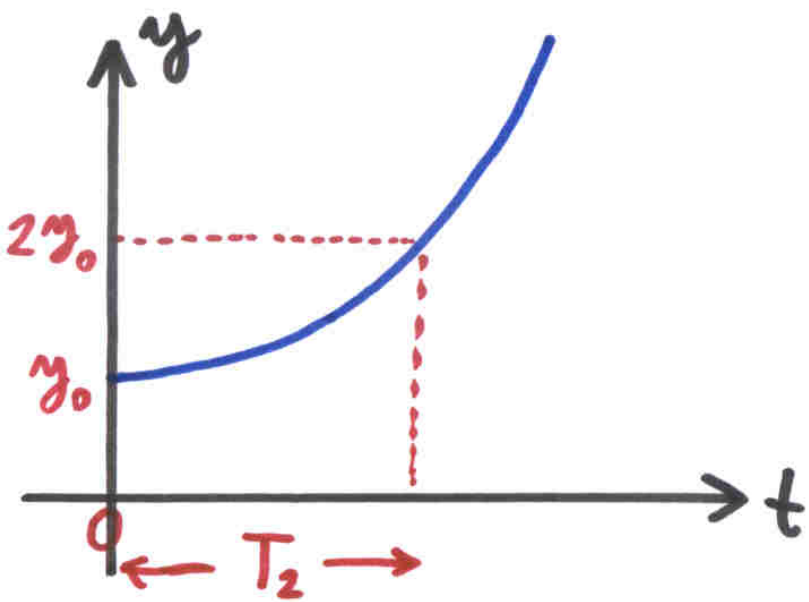
Let y = population size and y_0 = initial population size at t=0.

$$\frac{dy}{dt} = ry \Rightarrow r = \frac{\left(\frac{dy}{dt}\right)}{y}$$

$$y(t) = y_0 e^{rt}$$

Units of r : [r] = s^{-1}

Exponential Growth (r > 0)



- T_2 = Doubling time
- Sometimes we can estimate r > 0 from a measured doubling time:

$$2y_0 = y_0 e^{rT_2} \rightarrow$$

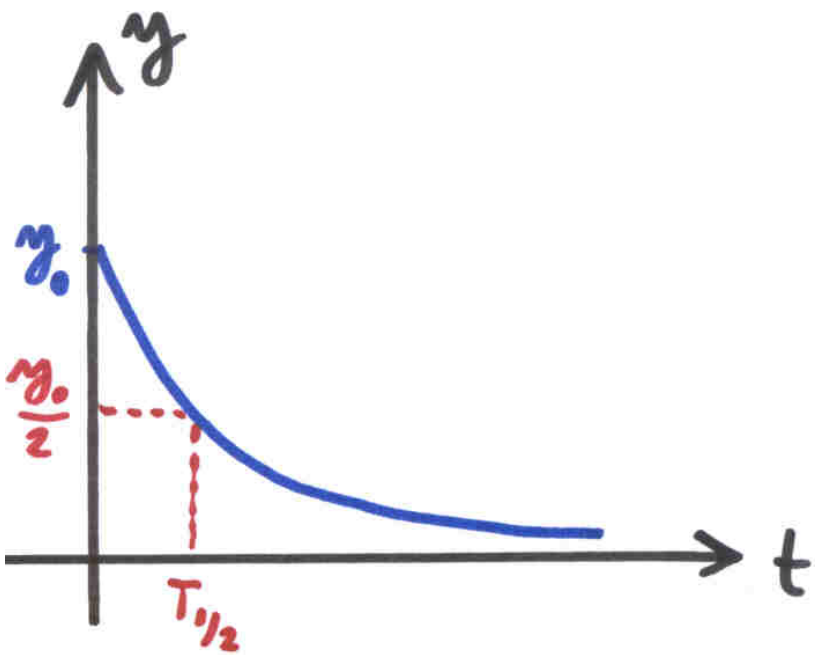
$$\rightarrow T_2 = \frac{\ln 2}{r} = \frac{0.693}{r} \leftrightarrow$$

$$\leftrightarrow r = \frac{\ln 2}{T_2}$$

Verification of a Malthusian Model

- Given <u>measurement data</u> (for instance Population vs. Time), how do we know if it is coming from a Malthusian model?
- One may plot “logarithm of the population” vs. Time, and check if it fits to a straight line [or do an exponential curve fitting]

Exponential Decay (r < 0)



- $T_{1/2}$="Half life"
- Sometimes we can estimate r < 0 from a measured half life:

$$\frac{1}{2}y_0 = y_0 e^{rT_2} \rightarrow$$

$$\rightarrow T_{1/2} = \frac{\ln 2}{(-r)} = \frac{0.693}{(-r)} \leftrightarrow$$

$$\leftrightarrow r = -\frac{\ln 2}{T_{1/2}}$$

Local Exponential Growth

Many populations exhibit exponential growth <u>for a while</u>

←→ “Local” (rather than “global”) validity

Local Malthusian Process

- When plotting log(population) vs. t you may detect at what value of t the curve begins to deviate from a straight line.
- Sometimes, the above curve may switch from one line to another.
- Eventually the line curves as t becomes large enough.

Environmentally Limited Population Growth

- In real populations there is no such a thing as a constant (per capita) growth rate r.
- Finite available resources eventually put an upper limit on population growth.

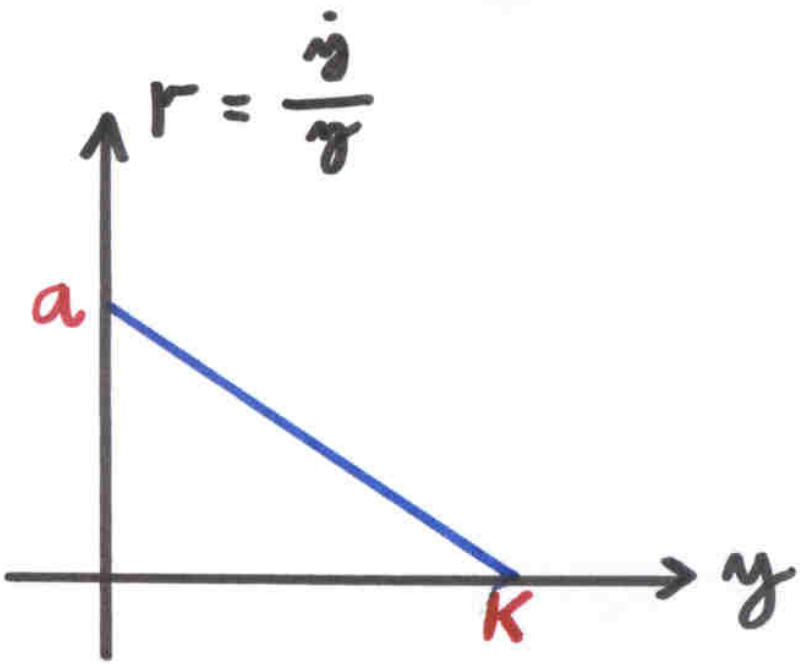
How do we model it mathematically?

Pierre-Francois Verhulst (1804-1849)

- Belgian mathematician
- Proposed a model that he named “logistic” (and it’s still a mystery why this name was used)

Verhulst's Logistic Model (1845)

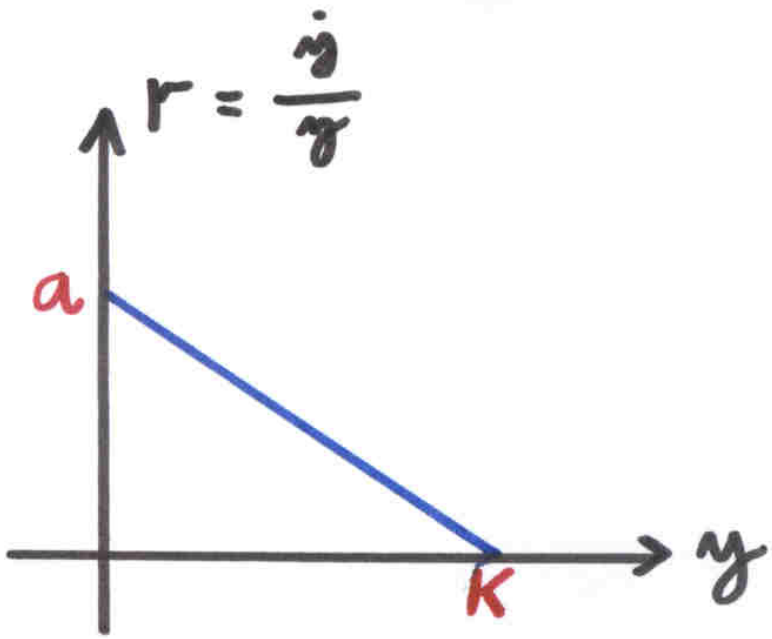
- <u>Assumption:</u> As population increases, the per-capita growth rate gets smaller linearly.



$$\boxed{r} = r(y) = \frac{\frac{dy}{dt}}{y} = \boxed{a(1-\frac{y}{K})}$$

Logistic Model: Carrying Capacity

- K = the <u>carrying capacity</u> of the population
- At y = K the growth rate becomes zero (r=0)



$$\boxed{r} = r(y) = \frac{\frac{dy}{dt}}{y} = \boxed{a(1-\frac{y}{K})}$$

Identifying a Logistic Model

- Given measurement data of Population vs. Time, how do we know if the data may come from a logistic model?
- Estimate r(t), using a finite difference approximation of a derivative.
- Such estimated curves are typically very noisy. Need to fit straight lines to the data, and check how good is the goodness-of-fit measure R

Logistic Model Solution

$$\frac{\frac{dy}{dt}}{y} = a(1-\frac{y}{K}) \Rightarrow$$

$$\Rightarrow \frac{dy}{dt} = ay(1-\frac{y}{K})$$

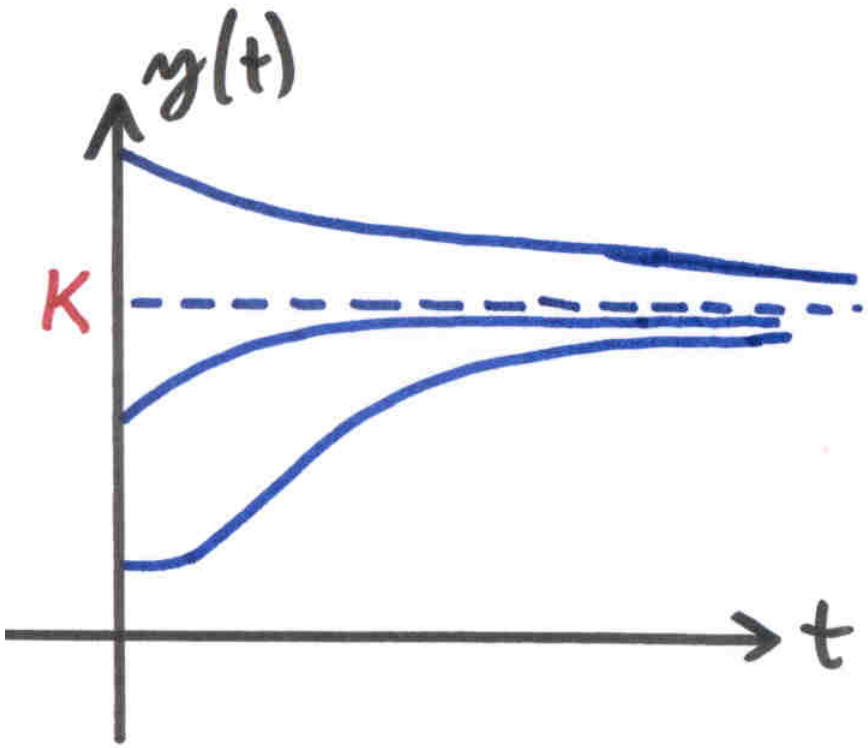
It is a nonlinear differential equation.
Analytic solution happens to be available:

$$y(t) = \frac{Ke^{at}y_0}{y_0(e^{at}-1)+K}$$

Behavior of Logistic Model Solution

$$y(t) = \frac{Ke^{at}y_0}{y_0(e^{at}-1)+K}$$

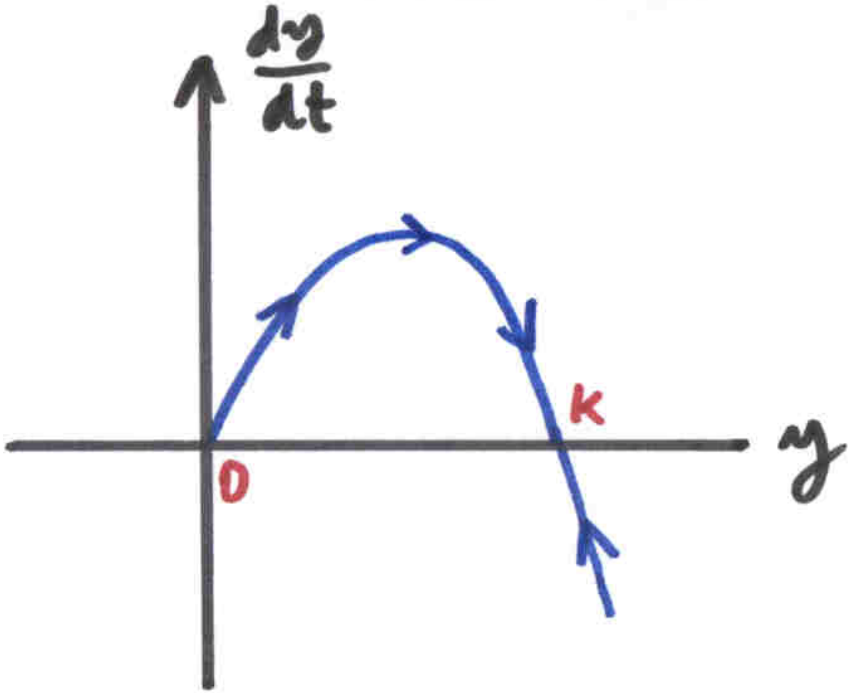
As t→∞ y→K, from any initial value



Logistic Model Phase-Plane Solution

$$\frac{dy}{dt} = ay(1-\frac{y}{K}) \Rightarrow$$

$$\Rightarrow \frac{dy}{dt} = 0 \Leftrightarrow y_e = 0 _ or _ y_e = K$$



y_e=0 is an unstable equilibrium point.

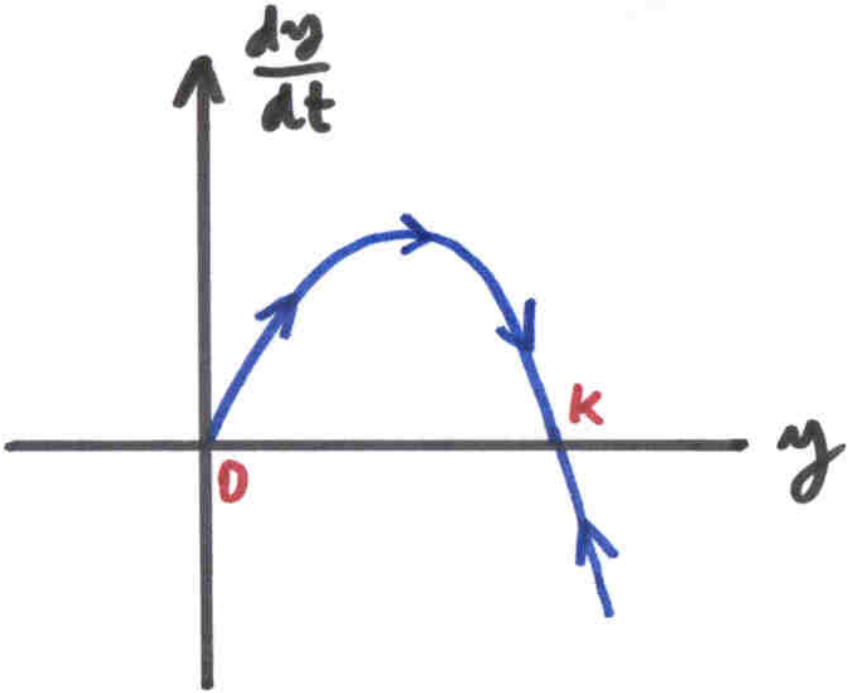
y_e=K is a stable equilibrium point.

Logistic Model Phase-Plane Solution – Time Parameter

$$\frac{dy}{dt} = ay(1 - \frac{y}{K}) \Rightarrow$$

$$\Rightarrow \frac{dy}{dt} = 0 \Leftrightarrow y_e = 0 _ or _ y_e = K$$

The time t is a parameter on the curve. Arrow indicates time growth.



Example:

- Census data for the State of Pennsylvania
- Example worked out by John Falcone (Florida Atlantic University) (Spring 2005).

Similar example: State of Pennsylvania Census Data

Year	Population
1790	434,373
1800	602,365
1810	810,091
1820	1,049,458
1830	1,348,233
1840	1,724,033
1850	2,311,786
1860	2,906,215
1870	3,521,951
1880	4,282,891
1890	5,258,113
1900	6,302,115
1910	7,665,111
1920	8,720,017
1930	9,631,350
1940	9,900,180
1950	10,498,012
1960	11,319,366
1970	11,793,909
1980	11,863,895
1990	11,881,643

Compute per-capita growth rate using finite difference approximation

$$r = \frac{\left(\frac{dy}{dt}\right)}{y} = \frac{dy}{(dt)(y)}$$

EXCEL formula for column C as =(B2-B1)/(B2*(A2-A1))

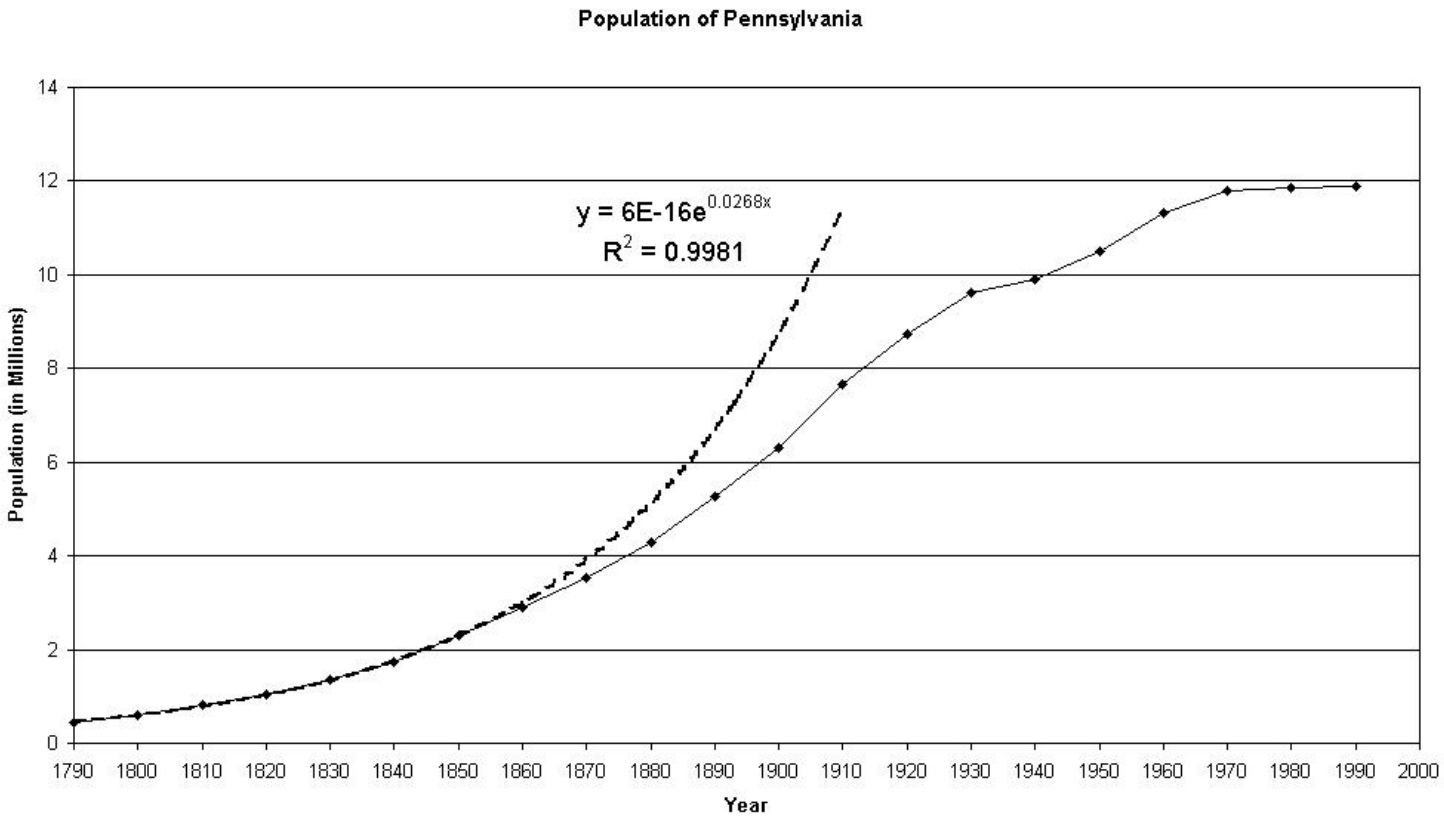
Per-capita growth rate

Year	Population	Per-Capita Growth Rate
1790	434,373	N/A
1800	602,365	0.02789
1810	810,091	0.02564
1820	1,049,458	0.02281
1830	1,348,233	0.02216
1840	1,724,033	0.02180
1850	2,311,786	0.02542
1860	2,906,215	0.02045
1870	3,521,951	0.01748
1880	4,282,891	0.01777
1890	5,258,113	0.01855
1900	6,302,115	0.01657
1910	7,665,111	0.01778

(Cont'd)

1920	8,720,017	0.01210
1930	9,631,350	0.00946
1940	9,900,180	0.00272
1950	10,498,012	0.00569
1960	11,319,366	0.00726
1970	11,793,909	0.00402
1980	11,863,895	0.00059
1990	11,881,643	0.00015

1790-1860


Population of Pennsylvania
y = 6E-16e^0.0268x
R^2 = 0.9981
Population (in Millions)
Year
0
2
4
6
8
10
12
14
1790
1800
1810
1820
1830
1840
1850
1860
1870
1880
1890
1900
1910
1920
1930
1940
1950
1960
1970
1980
1990
2000

Calculation of r

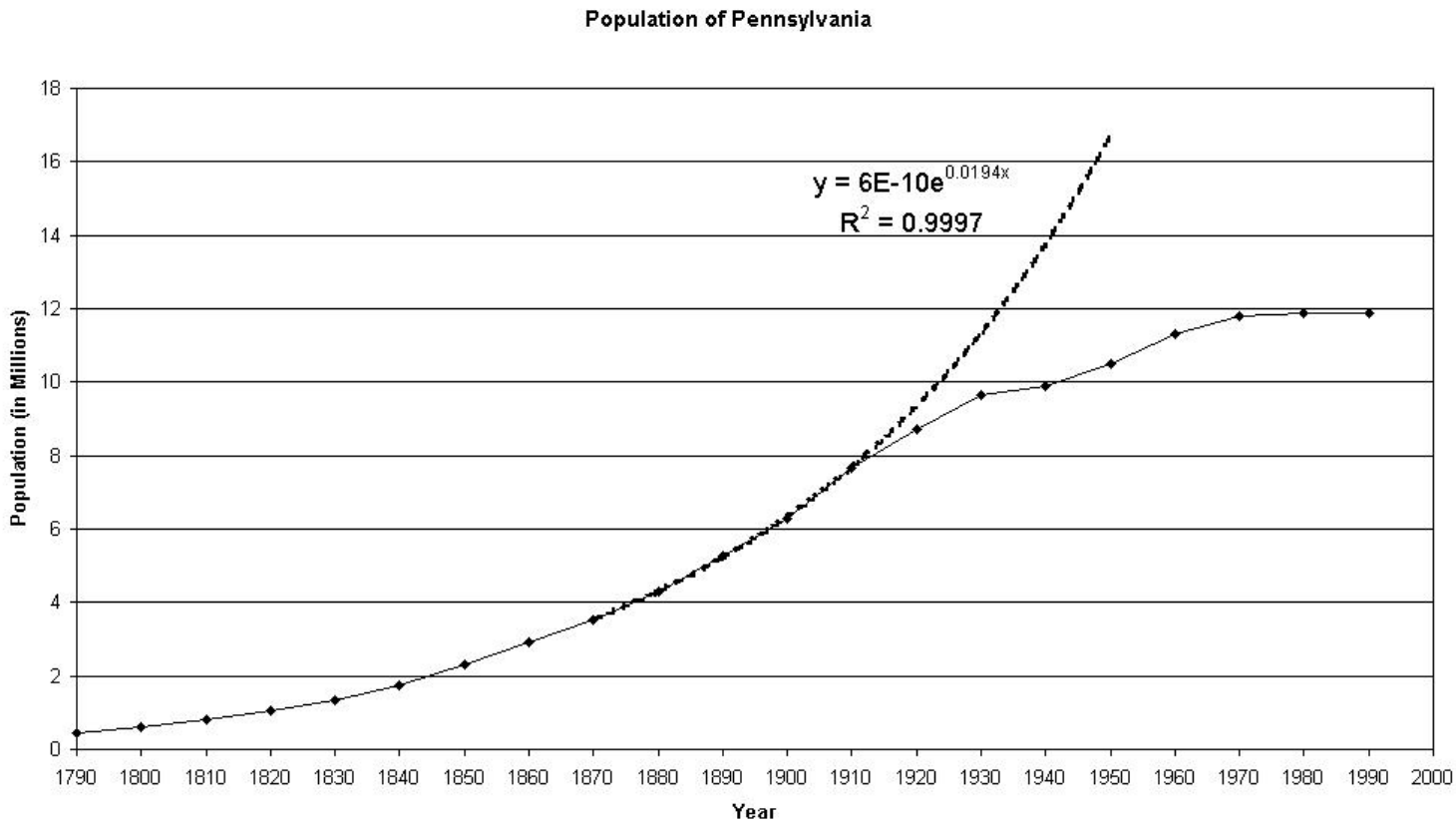
- We can observe from the R^2 values of the above curve that is indeed a good fit, and that the data exhibits local Malthusian growth in this region.
- If we wished to compute the r value from the data, we notice that the initial value in this region is 434,373. Given this, we know the doubling time is the time when the population hits 868,746. By inspection we estimate that this occurs around 1813, so the doubling time is 24 years.

Calculation of r

$$r = \frac{\ln 2}{T_2} = \frac{\ln 2}{24} = 0.0288$$

Comparing this value with that generated by EXCEL, 0.0268, shows that we are fairly close. The error can be attributed to estimation error.

1870-1910

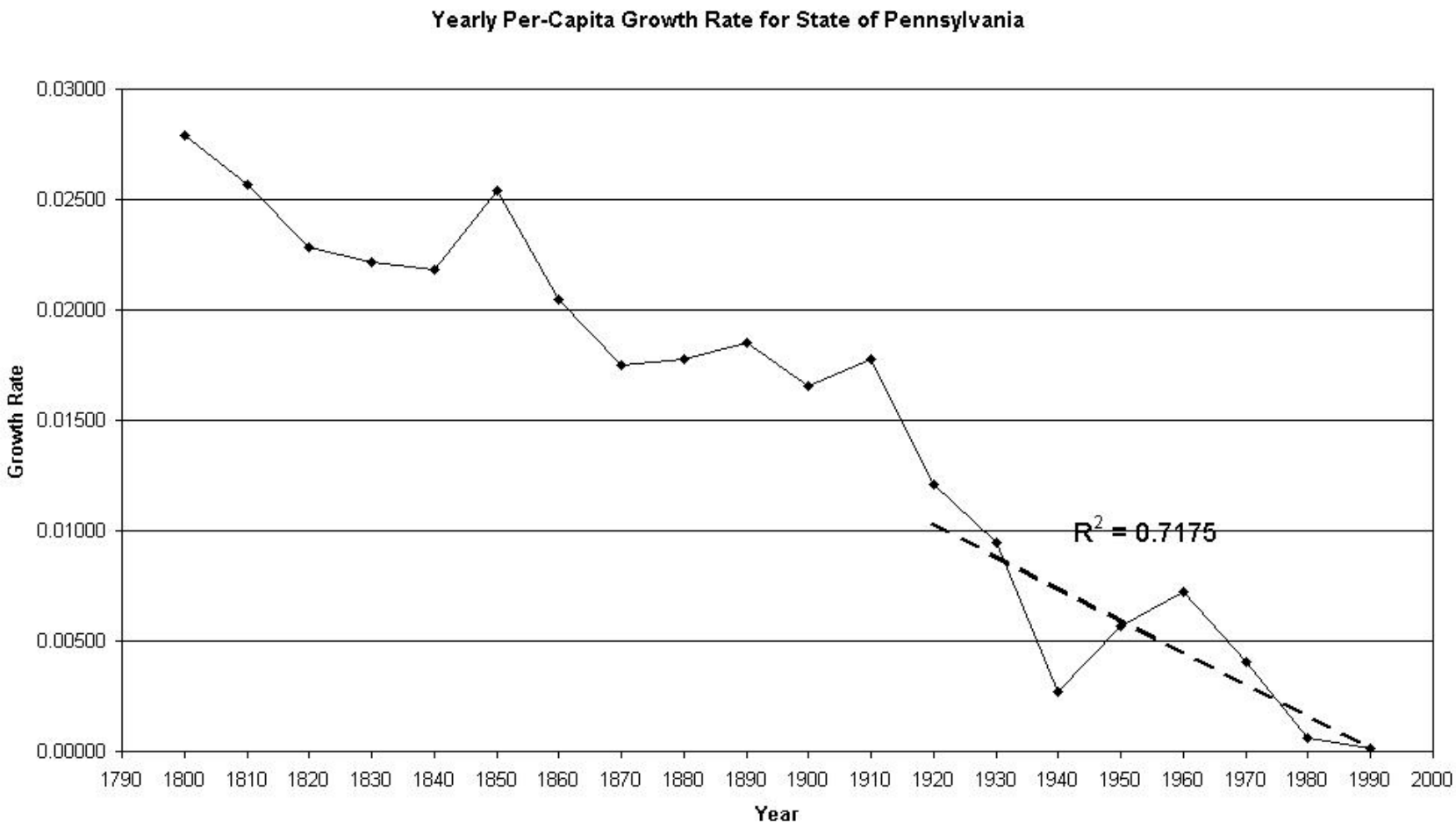

Population of Pennsylvania
y = 6E-10e^0.0194x
R^2 = 0.9997
Population (in Millions)
Year
0
2
4
6
8
10
12
14
16
18
1790
1800
1810
1820
1830
1840
1850
1860
1870
1880
1890
1900
1910
1920
1930
1940
1950
1960
1970
1980
1990
2000

We notice that the initial value in this region is 3,521,951. Given this, we know the doubling time is the time when the population hits 7,043,902. By inspection we estimate that this occurs around 1905, so the doubling time is 35 years.

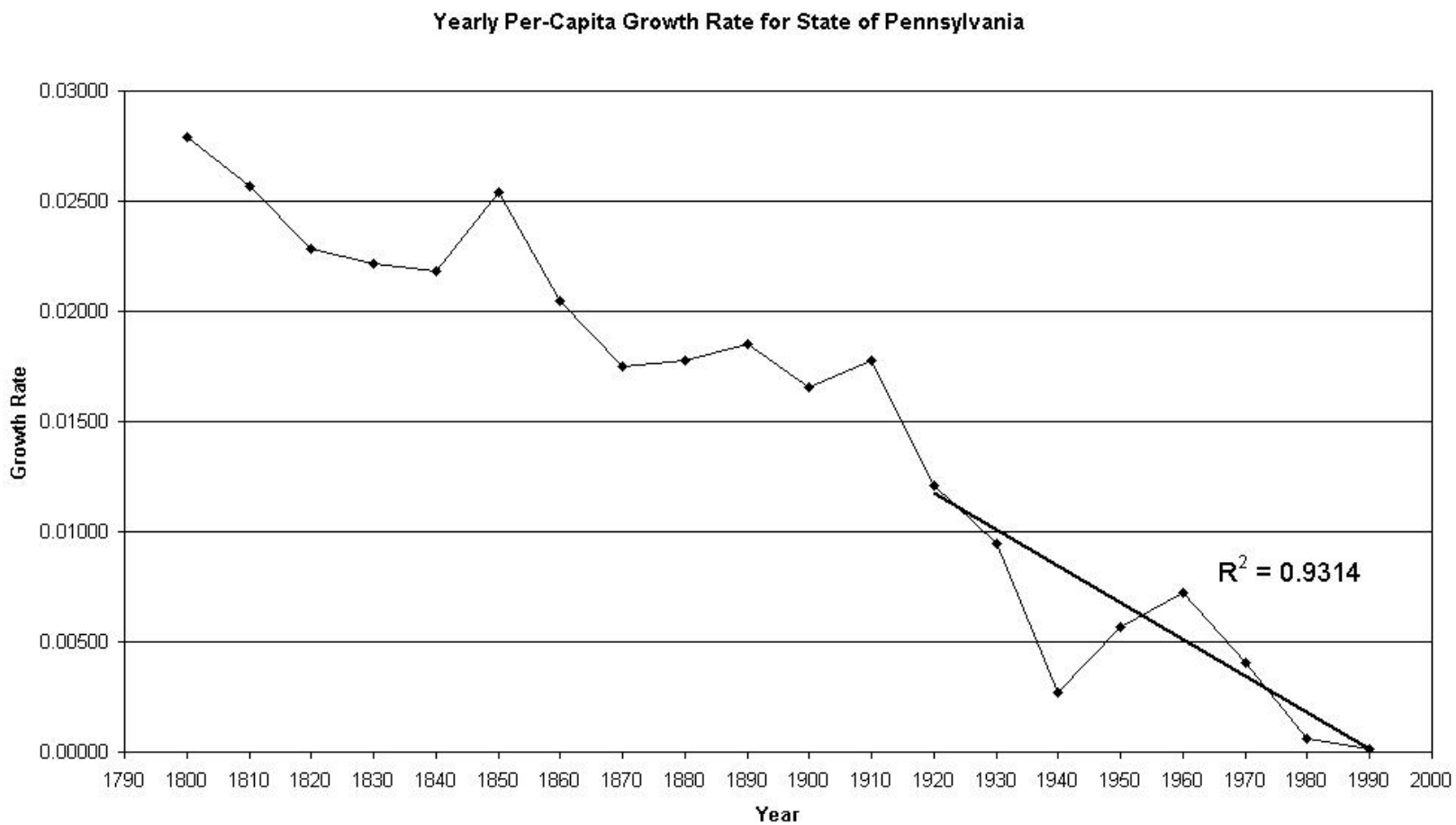
$$r = \frac{\ln 2}{T_2} = \frac{\ln 2}{35} = 0.0198$$

Comparing this value with that generated by EXCEL, 0.0194, shows that we are extremely close

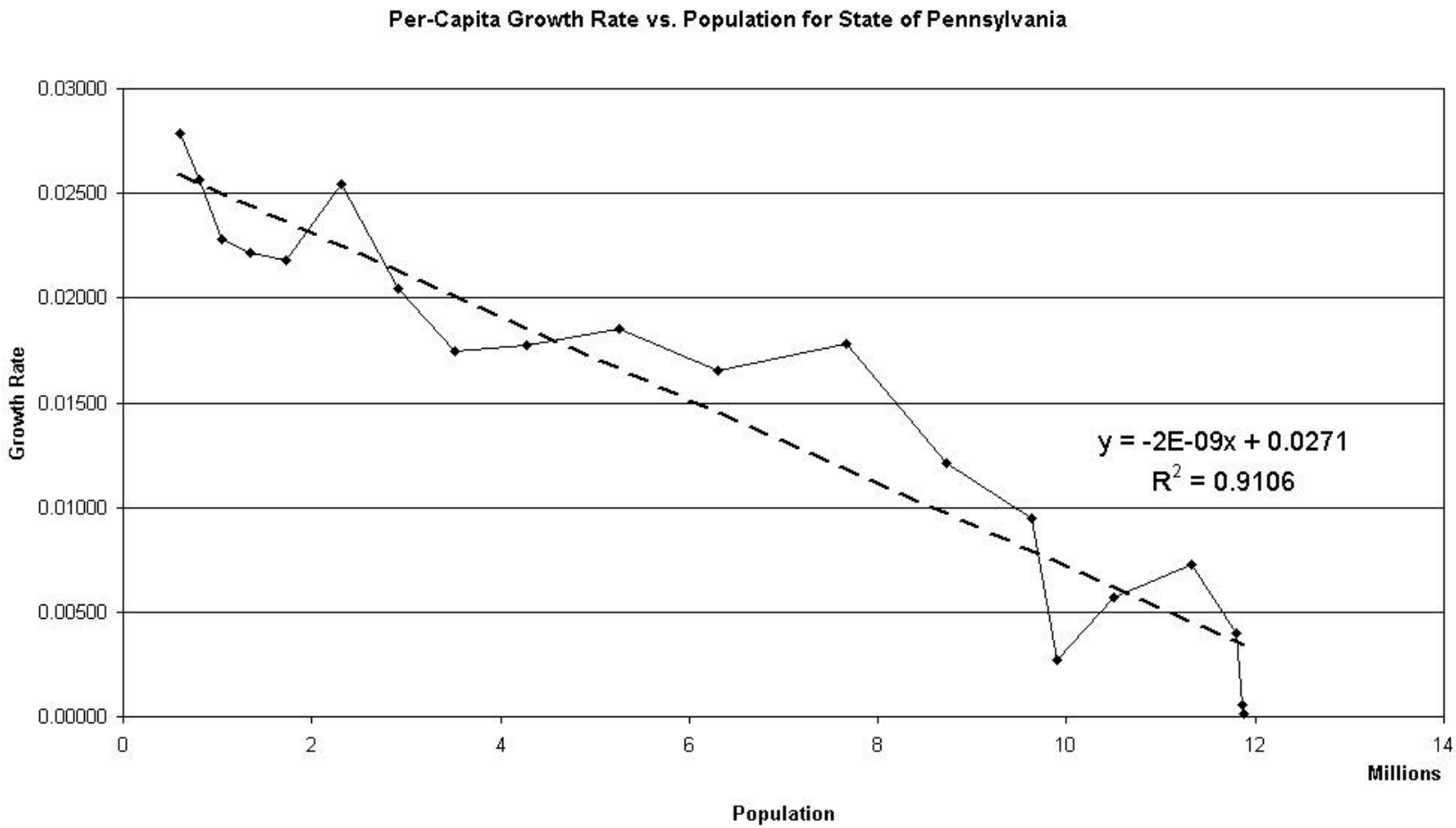
Logistic growth observed in the period 1920-1990



Same curve fit, removing 1940



Estimate the carrying capacity K by graphing the Per-Capita Growth Rate against the Total Population:



K calculation:

$$y = -.002x + 0.0271$$

$$0 = -.002x + 0.0271$$

$$x = \frac{-0.0271}{-0.002} = 13.55$$

Therefore K = 13.55M (although the final 3 data points seem to indicate otherwise!)