BME 5742 Bio-Systems Modeling and Control

Lecture 5

Extensions to the Logistic Model: Extinction and Predation

Extensions to the Logistic Model

- Logistic Model with Extinction
- Logistic Model with Predation

Extinction

Population may become extinct if size falls below a certain critical level:

- Predators eliminate the last few members
- Finding mates becomes more difficult
- Lack of genetic diversity → Increased susceptibility to epidemics.

How do we model "potential for extinction" mathematically?

"Predator Pit" Growth Rate Model (Extinction Threshold)

$$\boxed{r} = \frac{\frac{dy}{dt}}{y} = \boxed{a\left(\frac{y}{\theta} - 1\right)\left(1 - \frac{y}{K}\right)}$$

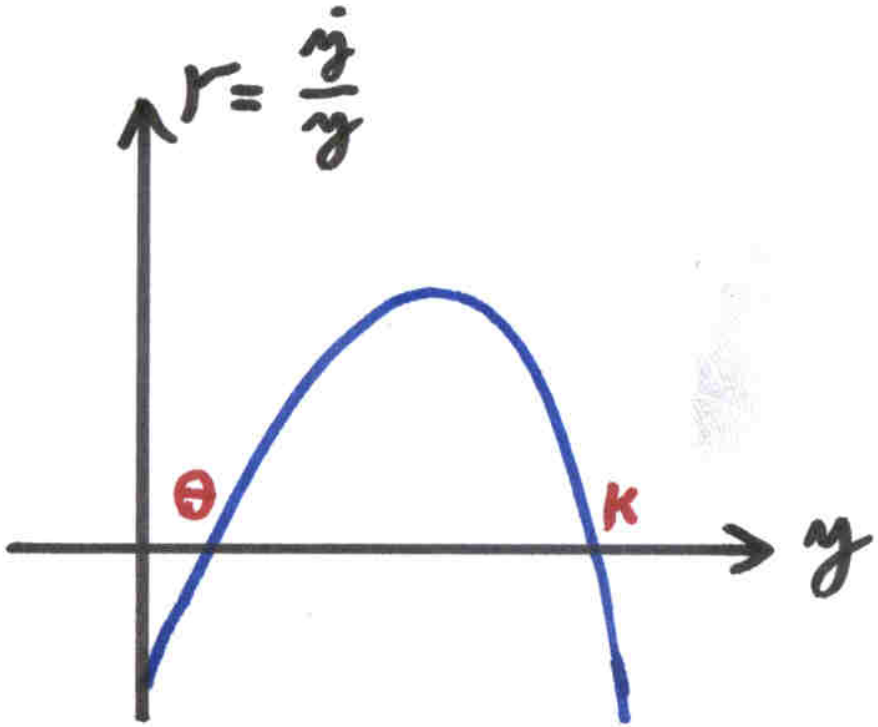
$$where_0 < \theta < K$$

Critical Size = θ, Carrying Capacity = K

Logistic Model with Extinction Threshold

$$r = \frac{\frac{dy}{dt}}{y} = a\left(\frac{y}{\theta} - 1\right)\left(1 - \frac{y}{K}\right)$$

$$where_0 < \theta < K$$

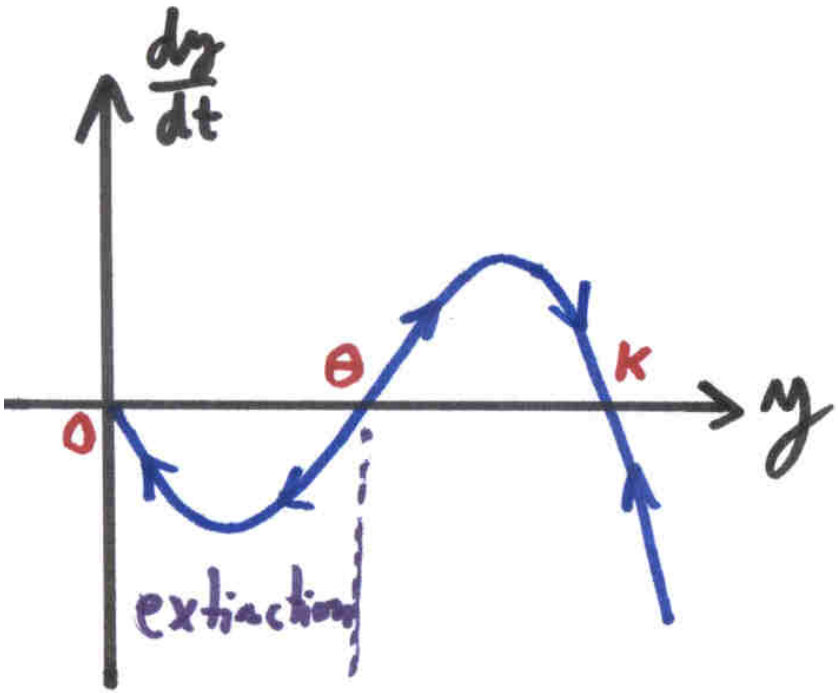


Growth rate is negative if
y<θ or if y>K

Logistic Model with Extinction Threshold – Phase-Plain Description

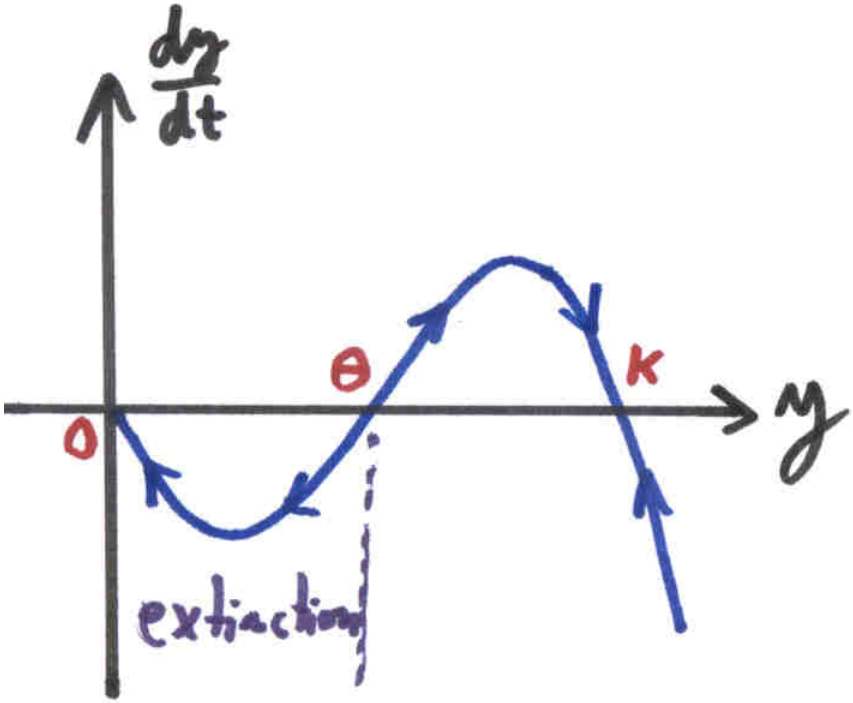
$$r = \frac{\frac{dy}{dt}}{y} = a(\frac{y}{\theta} - 1)(1 - \frac{y}{K})$$

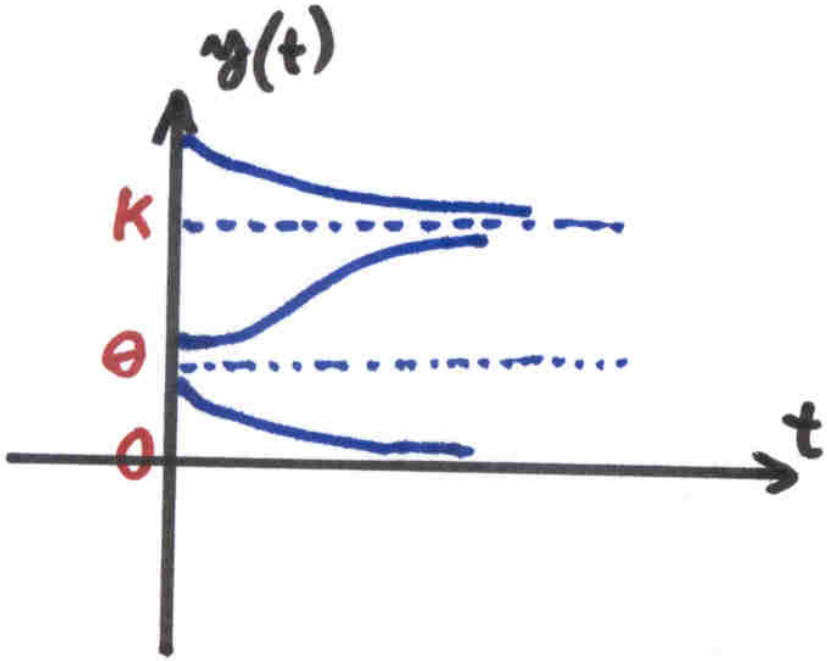
$$where_0 < \theta < K$$



Unstable equilibrium $y_e=\theta$
Stable equilibria $y_e=0$ and $y_e=K$

Logistic Model with Extinction Threshold – Sketch of Time Solution



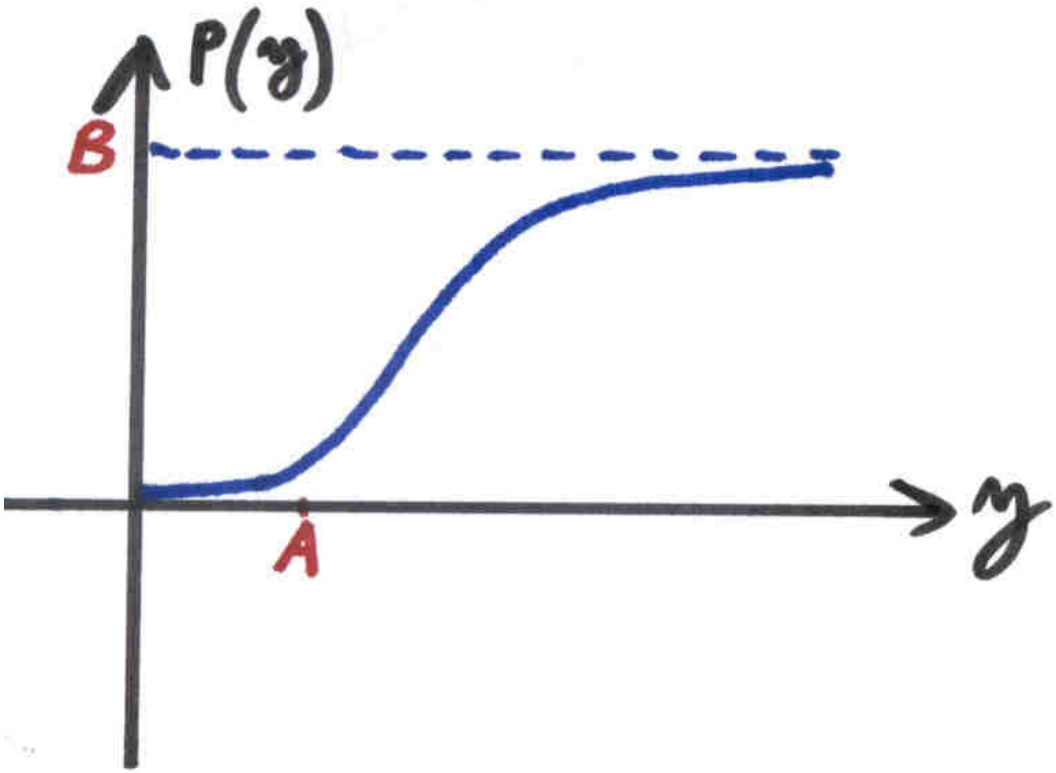


Logistic Growth Model with Predation

$$\frac{dy}{dt} = ay(1 - \frac{y}{K}) - p(y)$$

A reasonable functional form for p(y) should show a low rate of predation if y is sufficiently small, and limited rate of predation if y is large

Ludwig's Predation Model (1978)

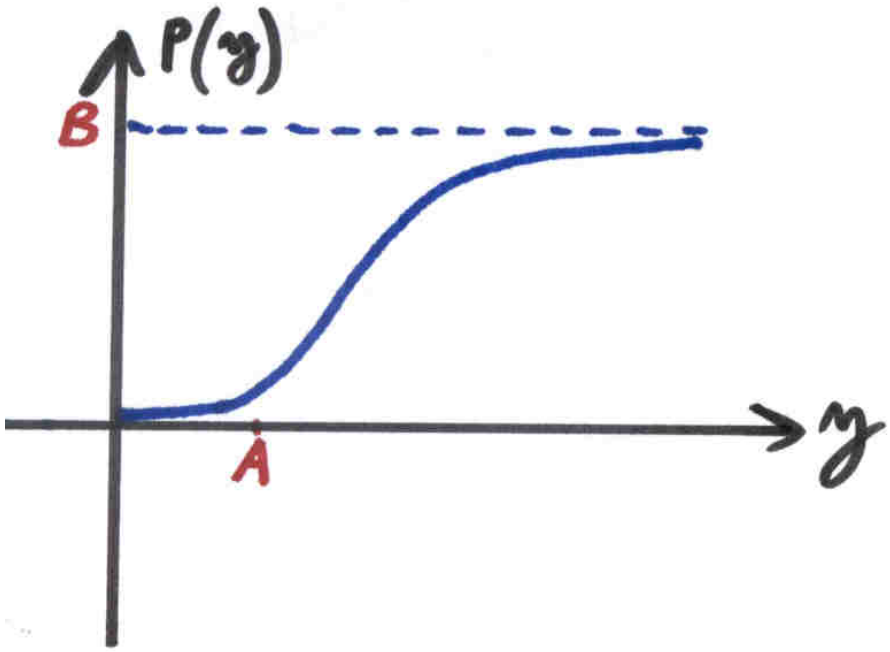


A = Predation threshold

B = Predation saturation

Ludwig's Predation Model – one of several plausible mathematical forms

$$p(y) = \frac{By^2}{A^2 + y^2}$$



Ludwig's mathematical model – Parameters units explained

- [A] = [K] = [y]
- [a] = time^{-1}
- [B] = [y]·(time)$^{-1}$
- It appears that 4 parameters a,K,A,B characterize the system; Actually, only two are needed!

$$p(y) = \frac{By^2}{A^2 + y^2}$$

$$\frac{dy}{dt} = ay(1 - \frac{y}{K}) - \frac{By^2}{A^2 + y^2}$$

The remaining slides may be skipped – material is not essential

It helps to express mathematical model equations in <u>scaled</u> and <u>non-dimensional</u> terms. Advantages are:

- Units become unimportant
- “Small” and “Large” have definite relative meaning
- Smaller number of relevant parameters are governing the solution behavior of the system

Scaling and Normalization is an art!!

Ludwig's Model Scaled and Normalized: Choice of New Variables

$$\frac{dy}{dt} = ay(1 - \frac{y}{K}) - \frac{By^2}{A^2 + y^2}$$

Define: (for the moment it may look like "black magic". We'll be more systematic later on)

$$u = \frac{y}{A} \qquad r = \frac{Aa}{B} \qquad q = \frac{K}{A} \qquad \tau = \frac{Bt}{A}$$

Ludwig's Model Scaled and Normalized: Substitution into Original Equation

$$\frac{dy}{dt} = ay(1-\frac{y}{K}) - \frac{By^2}{A^2+y^2}$$

$$u = \frac{y}{A} \qquad r = \frac{Aa}{B} \qquad q = \frac{K}{A} \qquad \tau = \frac{Bt}{A}$$

Then

$$d\tau = \frac{B}{A}dt \qquad \frac{du}{d\tau} = \frac{(\frac{du}{dt})}{(\frac{d\tau}{dt})} = \frac{\frac{1}{A}\cdot\frac{dy}{dt}}{\frac{B}{A}} =$$

$$= \frac{ay(1-\frac{y}{K}) - \frac{By^2}{A^2+y^2}}{B} = \frac{aAu(1-\frac{Au}{K}) - \frac{BA^2u^2}{A^2+A^2u^2}}{B} =$$

$$= ru(1-\frac{Au}{K}) - \frac{u^2}{1+u^2} = ru(1-\frac{u}{q}) - \frac{u^2}{1+u^2}$$

Ludwig's Model Scaled and Normalized

$$\frac{dy}{dt} = ay(1-\frac{y}{K}) - \frac{By^2}{A^2+y^2}$$

$$u = \frac{y}{A} \qquad r = \frac{Aa}{B} \qquad q = \frac{K}{A} \qquad \tau = \frac{Bt}{A}$$

Then

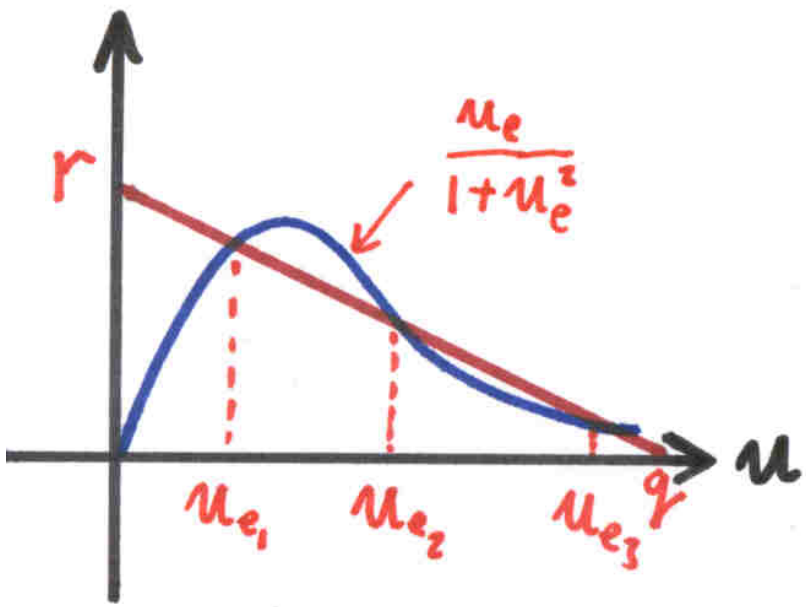
$$\frac{du}{d\tau} = ru(1-\frac{u}{q}) - \frac{u^2}{1+u^2}$$

All variables and parameters are non-dimensional

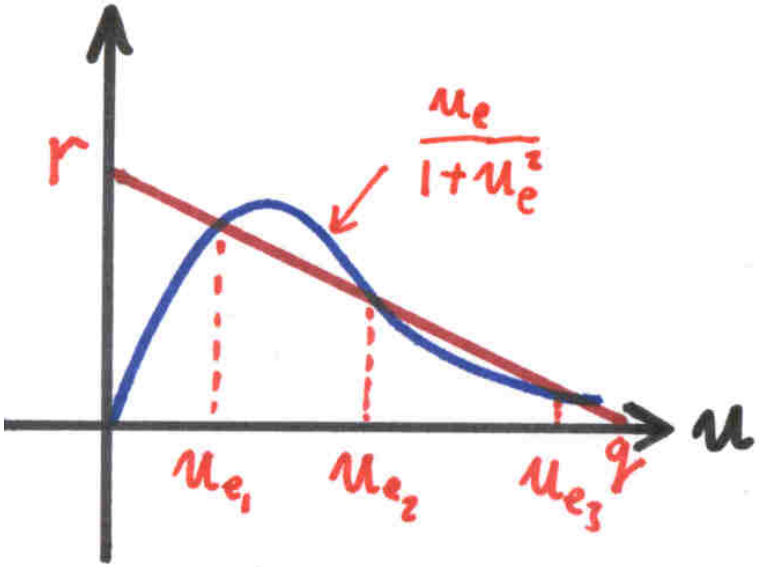
Equilibria of Ludwig's Model

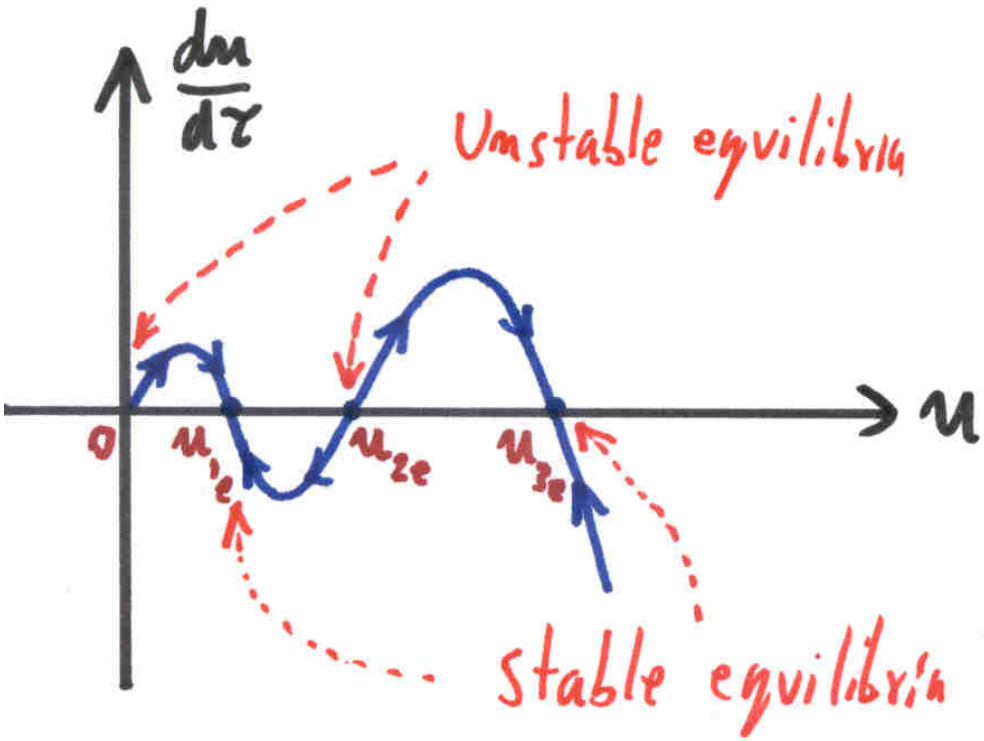
$$\frac{du}{d\tau} = 0 \Rightarrow ru_e(1 - \frac{u_e}{q}) = \frac{u_e^{\ 2}}{1 + u_e^{\ 2}} \Rightarrow$$

$$\Rightarrow u_e = 0 _ or _ r(1 - \frac{u_e}{q}) = \frac{u_e}{1 + u_e^{\ 2}}$$

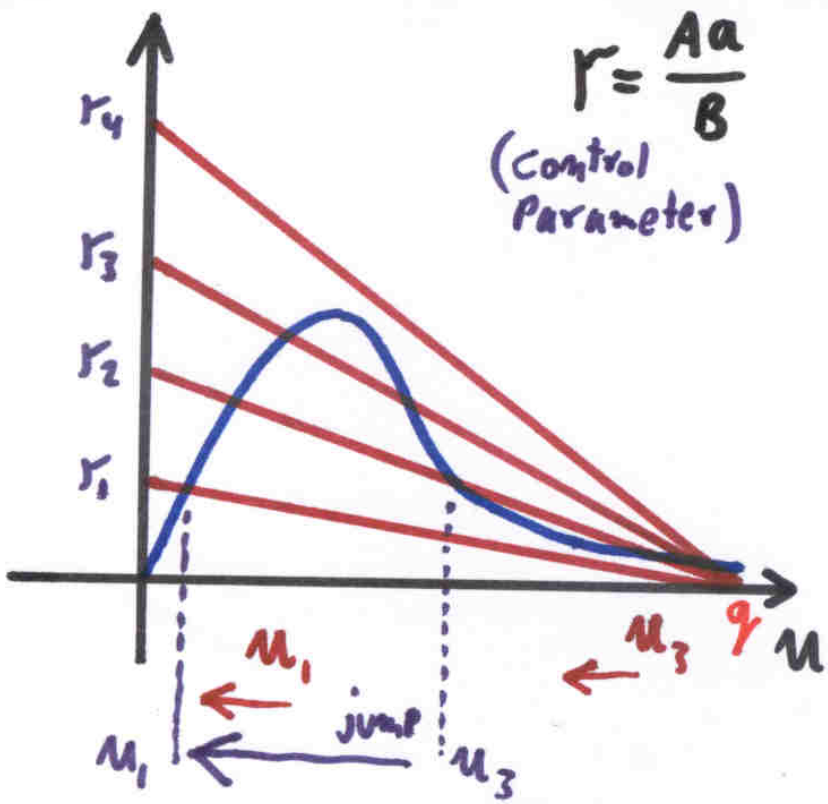


Equilibria Stability of Ludwig's Model





“Cusp Catastrophe” Phenomenon



- Number of equilibria depends on the parameters r,q
- “Refuge equilibrium” = u_{1e}
- “Outbreak equilibrium” = u_{3e}
- Application: Pest Control