BME 5742 Bio-Systems Modeling and Control

Lecture 8

Predator-Prey Models

An oscillating two-species model

Multi-Species Systems

- Single species models are unrealistic, as all species interact with some other species.
- Growth rate of each species depends on the amount of inter-species interaction.
- Simple model for the interaction of two species is a growth rate term proportional to the <u>product of the two population sizes</u>.

Classes of two-Species Models

- <u>Predator-Prey Model:</u> Growth rate interaction term of species A increases while that of species B decreases (and vice versa).
- <u>Competition Model:</u> Interaction causes both growth rates to decrease.
- <u>Mutualism (Symbiosis) Model:</u> Interaction causes both growth rates to increase.

Lotka-Volterra Predator-Prey Model (1925-1926)

$$\frac{dx}{dt} = rx - axy$$

$$\frac{dy}{dt} = -my + bxy$$

- Prey population = x
- Predator population = y
- Positive constants r,m,a,b.

Predator-Prey Model explained

$$\frac{dx}{dt} = rx - axy$$

$$\frac{dy}{dt} = -my + bxy$$

- Prey population = x
- Predator population = y
- With no interaction (a=b=0), x tends to increase, whereas y tends to decrease.
- Interaction decimates prey and enhances predator.

Prey Model Assumptions

- The prey, in the absence of any predation, grows unboundedly in a Malthusian way (this is the rx term).
- The effect of predation is to reduce prey's per capita growth rate by a term –axy proportional to both populations product.

Predator Model Assumptions

- The predator, in the absence of any prey, decays exponentially (this is the -mx term).
- The effect of predation is to enhance predator's per capita growth rate by a term +bxy proportional to both populations product.

Predator-Prey Model Equilibrium States

$$\frac{dx}{dt} = 0 = rx - axy$$

$$\frac{dy}{dt} = 0 = -my + bxy$$

There are two solutions x=y=0 and (x=m/b and y = r/a).

Model Solution at arbitrary initial conditions

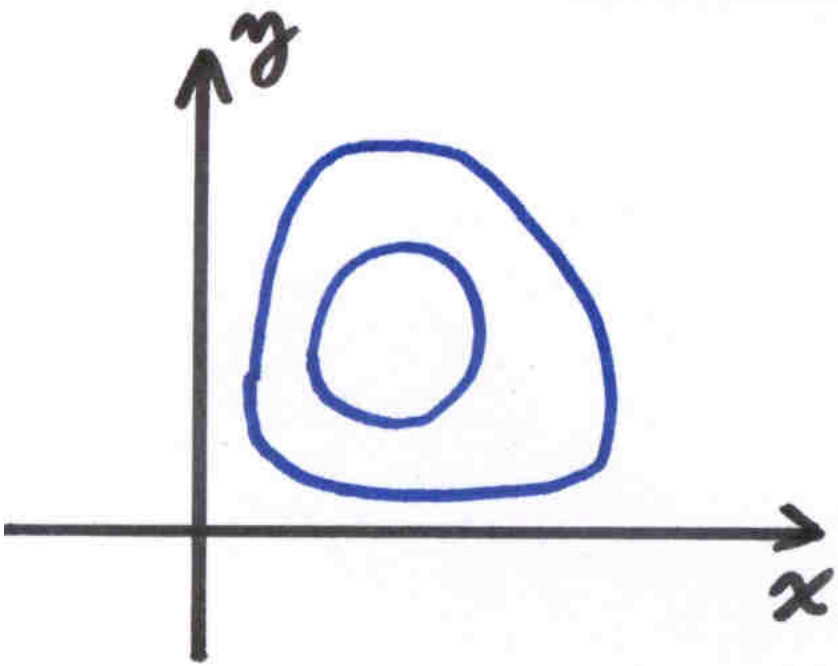
$x(0) = x_0 \neq m/b \qquad y(0) = y_0 \neq r/a$ (Simultaneously)

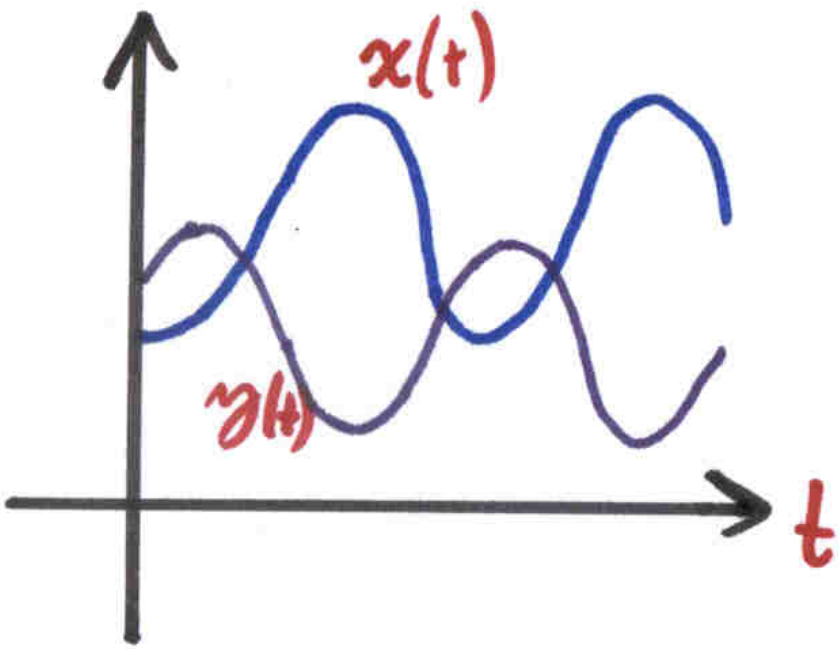
Then:

$$\frac{dy}{dx} = \frac{-my + bxy}{rx - axy}$$

These are closed curves in the x-y state-plane

Predator-Prey State-Plane and Time Solutions





Not a limit cycle! In a limit cycle the amplitude and frequency are constant regardless of initial conditions.

Normalization of the Predator-Prey Model

- The four parameter model with {r,m,a,b} reduces to a single-parameter model, if we scale time, and normalize each of the populations: 4-1-1-1=1
- Here is what seems to be a “tricky” normalizing:

$$u = \frac{b}{m} x \qquad v = \frac{a}{r} y \qquad \alpha = \frac{m}{r} \qquad \tau = r \cdot t$$

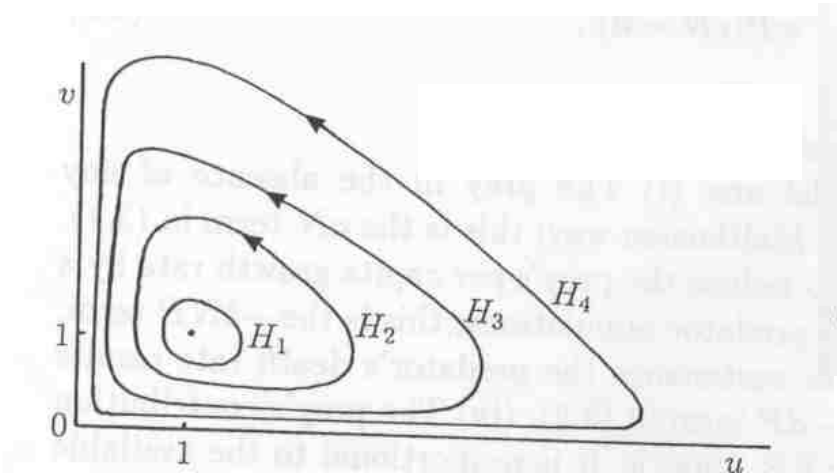
The resulting normalized Predator-Prey Model:

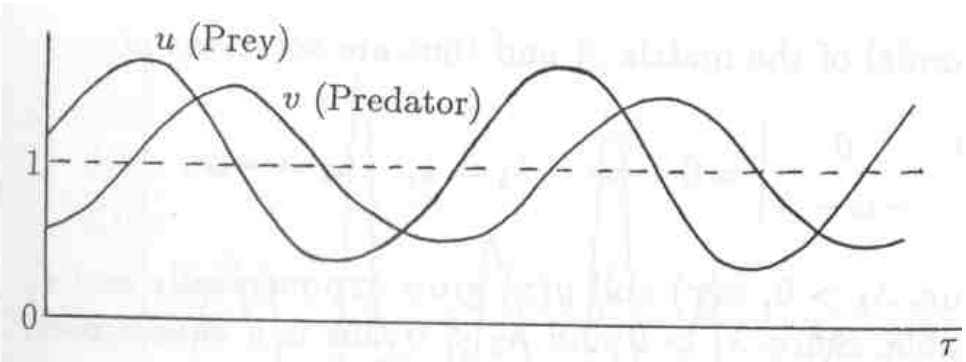
$$\frac{du}{d\tau} = u(1-v) \qquad \frac{dv}{d\tau} = \alpha v(u-1)$$

- Equilibrium states are u=v=0 and u=v=1

The resulting normalized Predator-Prey Model:

$$\frac{du}{d\tau} = u(1-v) \qquad \frac{dv}{d\tau} = \alpha v(u-1)$$





[Figures taken from Murray's book]

Meaning of the parameter α

- The single parameter α that controls the predator-prey model has the following meaning:
- Parameter $\alpha = m/r$ =

  (Predator's death rate)/(Prey's birth rate)
- The normalization is actually not that tricky. Here is a step-by-step explanation of the process:

Prey's equation normalization

$$\frac{dx}{dt} = x(r - ay) = rx(1 - \frac{a}{r}y) = rx(1 - v)$$

$[r]=(\text{time})^{-1}$; $[a]=[y]^{-1}(\text{time})^{-1}$

Therefore, ay/r is unit-less and we call it v (a normalized predator variable)

Time Scale implied by the Prey's equation

$$\frac{dx}{dt} = x(r - ay) = rx(1 - \frac{a}{r}y) = rx(1 - v)$$

Now we see that by a time scaling (shown below) we can get rid of the last parameter in the prey's equation.

$$\tau = rt \Leftrightarrow d\tau = rdt$$

Prey's equation time scaled

$$\frac{dx}{dt} = x(r - ay) = rx(1 - \frac{a}{r}y) = rx(1 - v)$$

$$\tau = rt \Leftrightarrow d\tau = rdt$$

$$\frac{dx}{d\tau} = \frac{dx}{dt} \cdot \frac{dt}{d\tau} = rx(1 - v) \cdot \frac{1}{r} = x(1 - v)$$

Predator's equation normalized

$$\frac{dy}{dt} = my(\frac{b}{m}x - 1) = my(u - 1)$$

$$u = \frac{b}{m}x$$

$$\frac{dy}{d\tau} = \frac{dy}{dt} \cdot \frac{dt}{d\tau} = \frac{m}{r}y(u - 1)$$

Finally: Substitution of variables

Predator-Prey oscillation solution

- A closed path in the u-v plane (or x-y plane) represents an oscillation. That is, x(t) and y(t) are periodic, having the same period.
- The predator-prey oscillation's features (amplitude, period, shape of time signals) vary as the initial values of x and y vary.
- This is not a <u>limit cycle</u> (a fixed oscillation to which the system converges from any initial conditions).

Extended Predator-Prey Models

More realistic models must include some logistic growth limitation. Here is a general model:

$$\frac{dx}{dt} = xF(x, y) \qquad \frac{dy}{dt} = yG(x, y)$$

Prey equation

Predator equation

How to make prey models more realistic:

$$\frac{dx}{dt} = xF(x, y)$$

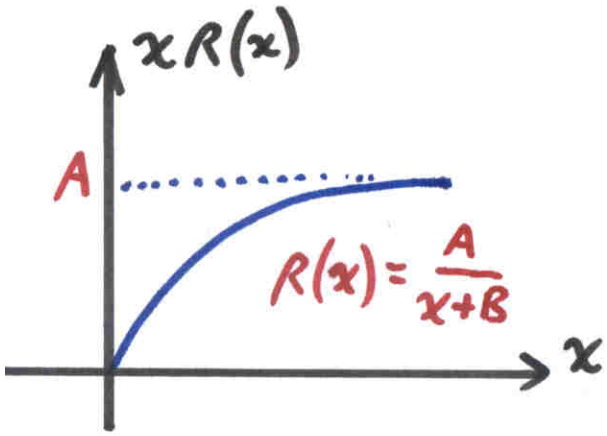
$$F(x, y) = r(1 - \frac{x}{K}) - yxR(x)$$

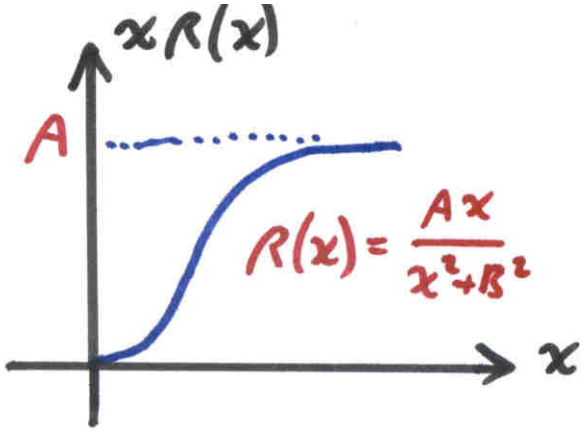
We include a logistic prey's growth model and a predation term. There are several possibilities for suitable predation models:

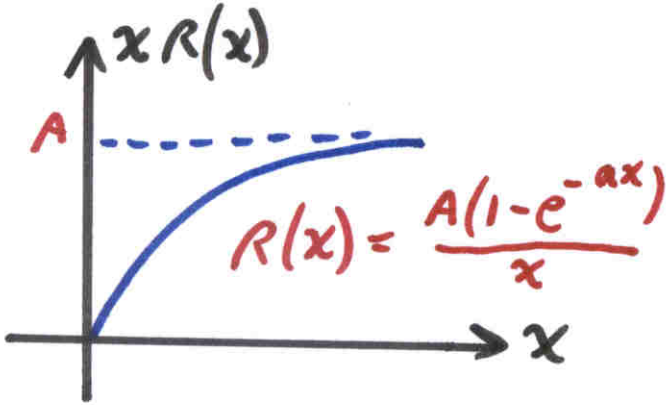
Prey Logistic Model with Predation

$$\frac{dx}{dt} = xF(x, y)$$

$$F(x, y) = r(1 - \frac{x}{K}) - yxR(x)$$





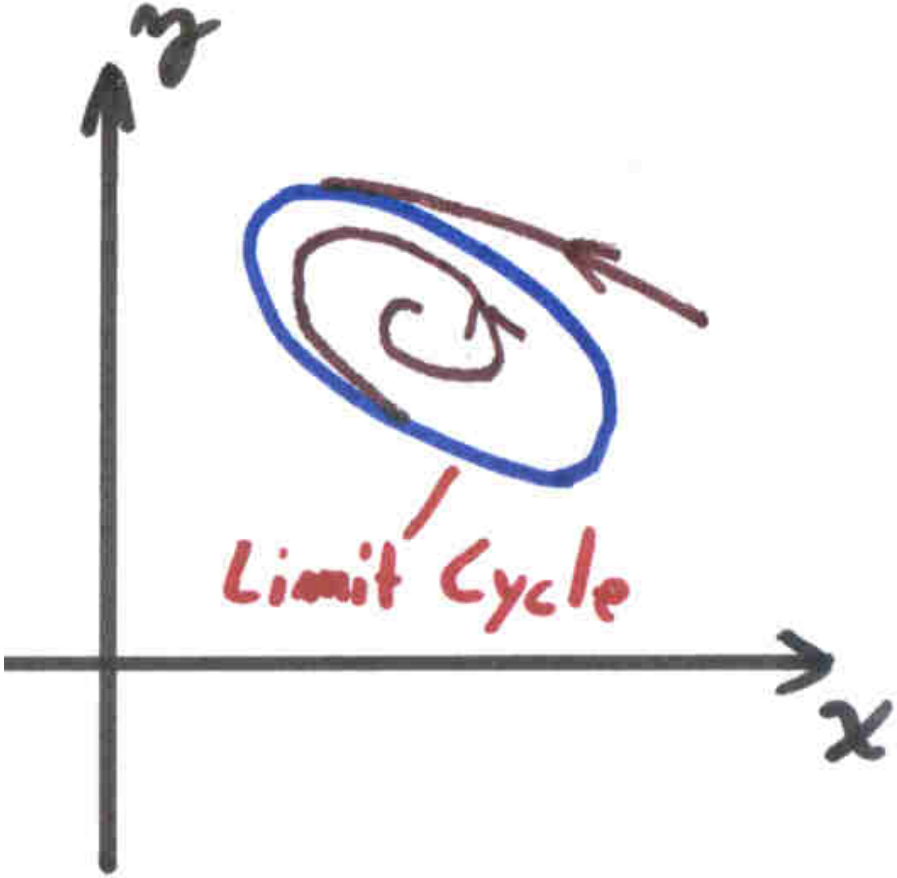


More realistic Predator Equation

$$\frac{dy}{dt} = yG(x, y)$$

More possibilities for a Predator Equation

$$\frac{dy}{dt} = yG(x, y)$$

G= -d+eR(x), where d and e are positive constants, and R(x) is as in the prey's equation.

Many of the extensions to the standard Lotka-Volterra model can lead to a stable limit cycle